1. Let $X$ be path connected, locally path connected, and semilocally simply connected. Let $H_0$ and $H_1$ be subgroups of $\pi_1(X, x_0)$ such that $H_0 \subset H_1$. Let $p_i: X_{H_i} \to X$ be covering spaces corresponding to the subgroups $H_i$. Prove that there is a covering space map $f: X_{H_0} \to X_{H_1}$ such that $p_1 \circ f = p_0$.

Let $y_i \in X_{H_i}$ be basepoints above $x_0$ ($i = 0, 1$). Then $p_{i*}(\pi_1(X_{H_i}, y_i)) = H_i$, and $H_0 \subset H_1$ implies there is a lift $f: X_{H_0} \to X_{H_1}$ of the map $p_0$, by the lifting criterion. That is, we have $f: X_{H_0} \to X_{H_1}$ such that $p_1 \circ f = p_0$.

We need to show that $X_{H_1}$ is evenly covered by $f$. Given $y \in X_{H_1}$ let $U \subset X$ be a path connected open neighborhood of $x = p_1(y)$ that is evenly covered by both $p_0$ and $p_1$. Let $V \subset X_{H_1}$ be the slice of $p_{1*}^{-1}(U)$ containing $y$. Note that $V$ is path connected, and so are all slices of $p_{1*}^{-1}(U)$ and $p_0^{-1}(U)$.

Denote the slices of $p_0^{-1}(U)$ by $\{W_z \mid z \in p_0^{-1}(x)\}$, where $W_z$ is the slice containing the point $z$. Let $C$ be the subcollection $\{W_z \mid z \in f^{-1}(y)\}$. Every slice $W_z$ is mapped by $f$ into a single slice of $p_{1*}^{-1}(U)$, since these sets are path connected. Since $f(z) \in p_1^{-1}(x)$, the image of $W_z$ is in $V$ if and only if $f(z) = y$. Hence $f^{-1}(V)$ is the union of the slices in $C$.

Given $W_z \in C$, the homeomorphisms $p_0|_{W_z}$ and $p_1|_{V \cap f}|_{W_z}$ are equal, and so $(p_1|_{V})^{-1} \circ p_0|_{W_z} = f|_{W_z}$. Hence the latter map is a homeomorphism, and $V$ is an evenly covered neighborhood of $y$.

2. Show that if a path connected, locally path connected space $X$ has finite fundamental group, then every map $X \to S^1$ is nullhomotopic. [Use the covering $\mathbb{R} \to S^1$.]

Let $f: X \to S^1$ be the map and $p: \mathbb{R} \to S^1$ the usual covering map. The image subgroup $f_*(\pi_1(X)) \subset \pi_1(S^1)$ is finite, and must therefore be trivial since $\mathbb{Z}$ has no non-trivial finite subgroups. Then $f_*(\pi_1(X)) \subset p_*(\pi_1(\mathbb{R}))$ (for any choice of basepoints), so the lifting criterion implies that there is a lift $\tilde{f}: X \to \mathbb{R}$. Let $F: X \times I \to \mathbb{R}$ be the straight line homotopy from $\tilde{f}$ to any constant map. Then $p \circ F$ is a homotopy from $f$ to a constant map, and $f$ is nullhomotopic.

3. Let $a$ and $b$ be the two free generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two $S^1$ summands. (a) Find the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by $\{a^2, b^2\}$. (b) Find the covering space corresponding to the normal subgroup generated by $\{a^2, b^2, (ab)^4\}$.

(A) The covering is shown below:

![Diagram of a covering space]

Each vertex is in the same orbit as its neighbor, via a covering translation given by rotation by $\pi$ in the circle through the two vertices. Thus, the covering group acts transitively on a fiber, and the covering is regular. Since $a^2$ and $b^2$ are loops in the cover, the corresponding normal subgroup...
of \( \langle a, b \rangle \) contains \( \langle a^2, b^2 \rangle \) (the smallest normal subgroup containing \( a^2 \) and \( b^2 \)). For the opposite inclusion, let \( T \) be the maximal tree given by all the leftward-oriented edges in the picture. Using this tree in the usual way, the free generators of the fundamental group are represented by the loops with labels \((ab)^k a^2 (ab)^{-k}\) or \((ab)^k b^{-2} a^{-1} (ab)^{-k}\) for \( k \in \mathbb{Z} \). These words are all conjugates in \( \langle a, b \rangle \) of \( a^2 \) and \( b^2 \), so the image subgroup is contained in \( \langle a^2, b^2 \rangle \).

(b) The covering is:

![Diagram](image)

As above, the covering is regular since the group of covering translations acts transitively on the vertices. (One of these is given by the composition of a reflection in the plane, and “inversion” which switches inside and outside edges.) Since \( a^2 \), \( b^2 \), and \( (ab)^4 \) are all represented by loops in the cover, the corresponding normal subgroup of \( \langle a, b \rangle \) contains \( \langle a^2, b^2, (ab)^4 \rangle \). Next let \( T \) be the maximal tree consisting of all inner edges except for one “b” edge. The fundamental group of the cover has 9 free generators, labeled by the words \( a^2 \), \( ab^2 a^{-1} \), \( aba(ab)^{-1} \), \( abab2(abab)^{-1} \), \( ababab2(ababab)^{-1} \), \( abababa(ababab)^{-1} \), and \( (ab)^4 \). These are all conjugates of \( a^2 \), \( b^2 \), and \( (ab)^4 \), so the subgroup is contained in \( \langle a^2, b^2, (ab)^4 \rangle \).

4. Find all connected 2-sheeted and 3-sheeted covering spaces of \( S^1 \vee S^1 \), up to isomorphism without basepoints.

There are three 2-sheeted coverings:

![Diagram](image)

There are seven 3-sheeted coverings:
5. Let \( \widetilde{X} \to X \) be a simply connected covering space of \( X \). Let \( A \subset X \) be path connected and locally path connected, and let \( \widetilde{A} \subset \widetilde{X} \) be a path component of \( p^{-1}(A) \). Show that the restricted map \( p : \widetilde{A} \to A \) is the covering space corresponding to the kernel of the homomorphism \( \pi_1(A) \to \pi_1(X) \).

We have seen in the previous course that \( \phi \circ j \) is a covering space map. It is easy to check that the restriction of a covering space to a path component is also a covering space. Now let \( x_0 \in A \) and \( y_0 \in \widetilde{A} \) be basepoints with \( p(y_0) = x_0 \).

Let \( i : A \to X \) and \( j : \widetilde{A} \to \widetilde{X} \) be the inclusion maps and let \( p' = p|_{\widetilde{A}} \). Then we have \( p \circ j = i \circ p' \), and hence \( p_* \circ j_* = i_* \circ p'_* \). Since \( p_* \circ j_* \) factors through \( \pi_1(\widetilde{X}, y_0) \), which is trivial, the composition \( i_* \circ p'_* \) is trivial. That is, \( p'_*(\pi_1(\widetilde{A}, y_0)) \subset \ker(i_*) \).

Next we show that \( p'_*(\pi_1(A, y_0)) \subset \ker(i_*) \). Let \( [f] \in \ker(i_*) \) where \( f \) is a loop in \( A \) at \( x_0 \). Let \( \tilde{f} \) be the unique lift of \( f \) to \( \widetilde{A} \) with initial endpoint \( y_0 \). Then \( j \circ \tilde{f} \) is a lift of \( i \circ f \) to \( \widetilde{X} \) with initial point \( y_0 \). Since \( [i \circ f] = 1 \) in \( \pi_1(X, x_0) \), this latter lift is in fact a loop in \( \widetilde{X} \). Hence \( \tilde{f} \) is a loop in \( \widetilde{A} \), and \( [f] = [p' \circ \tilde{f}] \) in \( \pi_1(A, x_0) \). That is, \( [f] \) is the image of \( [f] \) under \( p'_* \).

6. Let \( \phi : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation \( \phi(x, y) = (2x, y/2) \). This generates an action of \( \mathbb{Z} \) on \( X = \mathbb{R}^2 - \{0\} \). Show this action is a covering space action. Show the orbit space \( X/\mathbb{Z} \) is non-Hausdorff, and describe how it is a union of four subspaces homeomorphic to \( S^1 \times \mathbb{R} \), coming from the complementary components of the \( x \)-axis and the \( y \)-axis. Can you find the fundamental group of \( X/\mathbb{Z} \)?

First we need to show that every point has an open neighborhood that is disjoint from all of its translates under \( \phi \). For the point \((a, b)\) a product neighborhood of the form \((c, 2c) \times \mathbb{R} \) where \( a/2 < c < a \) (if \( a \neq 0 \)) or \( \mathbb{R} \times (d, 2d) \) where \( b/2 < d < b \) (if \( b \neq 0 \)) will work. Hence \( q : X \to X/\mathbb{Z} \) is a covering space map.

To see that \( X/\mathbb{Z} \) is not Hausdorff, consider two points of the form \((a, 0)\) and \((0, b)\) in \( X \). Any open neighborhood of \([a, 0]\) in \( X/\mathbb{Z} \) is given by \( \phi \)-invariant open neighborhood of the orbit of \((a, 0)\). Such a set must contain a small product neighborhood \( U = (a - \epsilon, a + \epsilon) \times (-\epsilon, \epsilon) \) and all of its translates under powers of \( \phi \). Similarly a neighborhood of \([0, b]\) is given by a \( \phi \)-invariant set containing \( V = (-\delta, \delta) \times (b - \delta, b + \delta) \) and its images under \( \phi^k \), \( k \in \mathbb{Z} \). Now take \( k \in \mathbb{Z} \) large enough that \( a/2^k \delta < \delta \) and \( 2^k \epsilon > b \). Then \( \phi^{-k}(U) \) intersects \( V \), and so \([a, 0]\) and \([0, b]\) cannot be separated by open sets in \( X/\mathbb{Z} \).

The group action preserves the subset \((0, \infty) \times \mathbb{R}\), and identifies each line \( x = a \) homeomorphically to the line \( x = 2a \). Hence the image of this set is homeomorphic to \( S^1 \times \mathbb{R} \). The same is true of the subsets \((-\infty, 0) \times \mathbb{R}, \mathbb{R} \times (0, \infty) \), and \( \mathbb{R} \times (-\infty, 0) \). Since these sets cover \( X \), the quotient is a union of four copies of \( S^1 \times \mathbb{R} \). Each line of the form \( \{a\} \times \mathbb{R} \) in the annulus \(((0, \infty) \times \mathbb{R})/\mathbb{Z} \) spirals around and limits onto two circles (the images of the two halves of the \( y \)-axis). Similarly, the circle \( S^1 \times \{0\} \) is the limiting circle for lines in two of the other annuli.
We know that $S$ has fundamental group $\mathbb{Z}$ and the group of covering translations of this (regular) covering is also $\mathbb{Z}$. Hence $\pi_1(X/\mathbb{Z})$ maps onto $\mathbb{Z}$ with kernel isomorphic to $\mathbb{Z}$. It follows that $\pi_1(X/\mathbb{Z})$ is a semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$. There are only two such groups, $\mathbb{Z} \times \mathbb{Z}$ and a non-abelian group (because there are only two automorphisms of $\mathbb{Z}$). To see that $\pi_1(X/\mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$, we show that its generators commute. These generators are given by loops in $X/\mathbb{Z}$ as follows: one is the image of the loop $\gamma$ in $X$ which generates $\pi_1(X,x_0)$; the other is the image of a path $\alpha$ in $X$ joining the basepoint $x_0$ to $\phi(x_0)$. It is not difficult to map $I \times I$ into $X$ so that its boundary maps to the path $\gamma \alpha \phi(\overline{\gamma}) \overline{\alpha}$. This is possible because $\phi$ preserves the orientation of $X$. Then the boundary of the image of this square in $X/\mathbb{Z}$ represents the commutator of the generators, and so they commute.