1. (a) Prove that if $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is an $n$-sheeted covering space then $p_\ast(\pi_1(\tilde{X}, \tilde{x}_0))$ is a subgroup of $\pi_1(X, x_0)$ of index $n$.

(b) Find two 2-sheeted covering spaces of the torus $S^1 \times S^1$ that are not isomorphic to each other (as covering spaces). Can you find three?

(A) We must assume (as usual) that $\tilde{X}$ is connected. Let $G = \pi_1(X, x_0)$ and $H = p_\ast(\pi_1(\tilde{X}, \tilde{x}_0))$. Write $p^{-1}(x_0)$ as $\{\tilde{x}_0, \ldots, \tilde{x}_{n-1}\}$. For any $[\gamma_1], [\gamma_2] \in G$ we claim that the cosets $[\gamma_1]H$ and $[\gamma_2]H$ are equal if and only if the lifts $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ starting at $\tilde{x}_0$ have the same endpoint $\tilde{x}_i$. Every $\tilde{x}_i$ can be joined to $\tilde{x}_0$ by a path $\gamma$ which represents some element $[\gamma] = [p \circ \tilde{\gamma}] \in G$. Hence the left cosets of $H$ are in one-to-one correspondence with $p^{-1}(x_0)$, and $[G: H] = n$.

To prove the claim, note that $[\gamma_1]H = [\gamma_2]H$ if and only if $[\gamma_1]^{-1}[\gamma_2]H = H$, i.e. if and only if $[\tilde{\gamma}_1^{-1}\tilde{\gamma}_2] \in H$. This occurs (by a theorem given in class) if and only if $\tilde{\gamma}_1 \ast \tilde{\gamma}_2$ lifts to a loop in $\tilde{X}$. Hence if $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have the same endpoint, $[\gamma_1]H = [\gamma_2]H$. Conversely, if $\tilde{\gamma}_1(1) \neq \tilde{\gamma}_2(1)$ then $\tilde{\gamma}_1 \ast \tilde{\gamma}_2$ is a lift of $\tilde{\gamma}_1 \ast \tilde{\gamma}_2$ that is not a loop, and so $[\gamma_1]H \neq [\gamma_2]H$.

(b) The first covering is $p_1: S^1 \times S^1 \to S^1 \times S^1$ given by $(z, w) \mapsto (z^2, w)$ in complex notation. The image subgroup in the fundamental group of $S^1 \times S^1$ is $2\mathbb{Z} \times \mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z}$. The second covering is $p_2: S^1 \times S^1 \to S^1 \times S^1$ given by $(z, w) \mapsto (z, w^2)$ with image subgroup $\mathbb{Z} \times 2\mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z}$. These subgroups are not conjugate (which is the same as being equal, in an abelian group), and so there is no choice of basepoints making the coverings isomorphic.

There is a third covering space corresponding to the subgroup $\{(m, n) \mid m + n \text{ is even}\}$. This is the kernel of the homomorphism $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ where each generator goes to the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$. There are no other 2-sheeted covers.

2. (a) Find all covering spaces of the circle up to covering space isomorphism.

(b) Find all homomorphisms between these covering spaces.

(c) If you ignore the covering maps to $S^1$, what are the covering spaces of $S^1$ up to homeomorphism? That is, which spaces arise in part (a)?

(A) The subgroups of $\mathbb{Z} = \pi_1(S^1, 1)$ are $n\mathbb{Z}$ for positive integers $n$, and the trivial group. The universal cover is $p: \mathbb{R} \to S^1$ given by $t \mapsto e^{it}$. There are also coverings $p_n: S^1 \to S^1$ given by $e^{it} \mapsto e^{int}$ (that is, $z \mapsto z^n$). We have seen in homework that the image subgroup $p_n(\pi_1(S^1, 1))$ is $n\mathbb{Z} \subset \mathbb{Z}$. Note that the coverings $p_n$ and $p_m$ are not isomorphic for $n \neq m$ since the subgroups are not equal (or alternatively, they have different numbers of sheets). We have found all coverings because they give all subgroups of $\mathbb{Z}$.

(b) First, since the trivial subgroup is contained in every subgroup, there exist homomorphisms from the universal cover to every cover. For the case $p_n: S^1 \to S^1$ the homomorphism $f: \mathbb{R} \to S^1$ is $t \mapsto e^{it/n}$, since $p_n(f(t)) = (e^{it/n})^n = e^{it}$. Next, there is a homomorphism from $p_n: S^1 \to S^1$ to $p_m: S^1 \to S^1$ if and only if $n\mathbb{Z} \subset m\mathbb{Z}$, i.e. if and only if $n$ divides $m$. When this occurs, say $n = km$, the homomorphism $f: S^1 \to S^1$ is given by $z \mapsto z^k$, since $p_m(f(z)) = (z^k)^m = z^n = p_n(z)$.
(c) The covering spaces are $S^1$ and $\mathbb{R}$.

3. (a,b,c) Same as question 2, but for covering spaces of the Möbius band. The Möbius band is the quotient of the square $I \times I$ by the equivalence relation that identifies the point $(0, t)$ to the point $(1, 1-t)$ for each $t \in I$.

Here is an alternate description of the Möbius band that may be useful. Let $f: \mathbb{R} \times I \to \mathbb{R} \times I$ be the map $(x, t) \mapsto (x+1, 1-t)$. Then the Möbius band is the quotient of $\mathbb{R} \times I$ by the equivalence relation whose equivalence classes are the orbits of $f$. That is, $(x, t) \sim (x', t')$ if and only if $(x', t') = f^k(x, t)$ for some $k \in \mathbb{Z}$.

(A) Let $M$ be the Möbius band. The “core circle” $C \subset M$ (the image of $I \times \frac{1}{2}$ in the first description, or of $\mathbb{R} \times \frac{1}{2}$ in the second) is a deformation retract of $M$, so $\pi_1(M) \cong \mathbb{Z}$. Therefore there exist a universal cover and covers for each positive integer $n$.

The universal cover is $\mathbb{R} \times I$ mapping by $(x, t) \mapsto [(x, t)]$. One can check that every point $(x, t) \in \mathbb{R} \times I$ has a neighborhood $U$ such that $f^k(U)$ is disjoint from $U$ for every $k \neq 0$. Then, by the definition of the equivalence relation, $U$ maps homeomorphically onto its image in $M$, and the preimage of this set is simply the disjoint union $\bigcup_k f^k(U)$, with each component mapping by a homeomorphism. That is, the image of $U$ is evenly covered by the map, and hence this is a covering space.

We know that there is a unique covering space with $n$ sheets for every integer $n$. To construct it we use the fact that the covering map above factors through it, and so it is a quotient space which lies between $\mathbb{R} \times I$ and $M$. We define $X_n$ to be the quotient of $\mathbb{R} \times I$ by the relation $(x, t) \sim (x', t')$ if $(x', t') = f^{kn}(x, t)$ for some $k \in \mathbb{Z}$. The covering map $p_n : X_n \to M$ is the natural map $[(x, t)]_n \mapsto [(x, t)]$ (remember, these are two different equivalence relations, but the former is finer than the latter, so the map is well defined). The verification that this is a covering map is similar to the argument given above.

Here is an alternate description of $X_n$: it is the quotient of $[0, n] \times I$ by the equivalence relation that identifies the point $(0, t)$ to the point $(n, g^n(t))$ for each $t \in I$, where $g(t) = 1-t$. The map $p_n$ takes the subspace $[i, i+1] \times I \subset X_n$ to $M$ by $(x, t) \mapsto (x-i, g^i(t))$. One can check that this respects the edge-identifications and hence is well defined.

(b) As with problem 2, there is a homomorphism $X_n \to X_m$ if and only if $m$ divides $n$, and there is always a homomorphism $\mathbb{R} \times I \to X_n$. This latter homomorphism is simply given by $(x, t) \mapsto [(x, t)]_n$. For the others, if $n = km$ then the map of equivalence classes $[(x, t)]_m \mapsto [(x, t)]_n$ is well defined: if $(x, t) \sim_m (x', t')$ then $(x', t') = f^{jn}(x, t)$ for some $j$, hence $(x', t') = f^j f^{km}(x, t)$, so $(x, t) \sim_n (x', t')$. It is clear that this map is compatible with the covering maps to $M$.

(c) The covering spaces are homeomorphic to $M$ (n odd), $S^1 \times I$ (n even), and $\mathbb{R} \times I$. This is seen from the second description of $X_n$, which is a strip with opposite sides identified, with coherent or incoherent orientations, according to the parity of $n$.

4. Let $X$ be the 3-fold dunce cap. That is, $X$ is a 2-dimensional cell complex with one 0-cell, one 1-cell, and one 2-cell, whose attaching map $S^1 \to X^1$ winds 3 times around (note that $X^1$ is a circle). Describe a simply connected covering space of $X$ and its covering map to $X$. [Hint: it is a cell complex having 3 i-cells for $i = 0, 1, 2$.]

For a picture of $X$, see page 52 of Hatcher. This shows a neighborhood of the 1-skeleton; just attach a disk to the boundary circle to get $X$. The fundamental group of $X$ is $\mathbb{Z}/3\mathbb{Z}$. Let $p : \tilde{X} \to X$
be the universal cover (which we are trying to construct). It is a 3-sheeted cover by problem 1, since the trivial subgroup has index 3 in the fundamental group.

The picture from Hatcher gives a way of understanding \( \tilde{X} \). In the part shown, if you travel once around the “equator” circle, the three “fins” will be permuted by a one-third rotation. In \( \tilde{X} \), this neighborhood of the 1-skeleton will unwind in a three-to-one fashion, and in this cover, the “fins” will make three one-third rotations before joining up again; that is, up to homeomorphism, this neighborhood is topologically a product (of the “tripod” with the circle). The disk that is missing from the picture will be covered by three disks, attached to the three boundary circles.

So \( \tilde{X} \) is a circle (made of three 0-cells and three 1-cells) with three 2-cells attached, each by a homeomorphism \( S^1 \to \tilde{X}^1 \).

The map to \( X \) is given as follows. Consider \( \tilde{X} \) as three closed disks \( D_1 \cup D_2 \cup D_3 \) with boundary circles identified (by the identity map). Consider \( X \) as a single disk \( D \) with identifications made on its boundary (i.e. \( z \sim z \cdot e^{2\pi i/3} \) for all \( z \in S^1 \)). For \( x \in D_1 \) map \( x \) to \( x \in D \). For \( x \in D_2 \) map \( x \) to \( r(x) \in D \) where \( r: D \to D \) is rotation by \( 2\pi/3 \) (or \( r(x) = x \cdot e^{2\pi i/3} \) if \( x \) is considered as a complex number). For \( x \in D_3 \) map \( x \) to \( r^2(x) = x \cdot e^{4\pi i/3} \). This is continuous as a map \( D_1 \cup D_2 \cup D_3 \to X \) and is constant on equivalence classes, hence gives a continuous map \( \tilde{X} \to X \). The interior of \( D \) is evenly covered since \( r: D \to D \) is a homeomorphism. To check that points on the 1-skeleton have evenly covered neighborhoods, one should carefully draw pictures of these neighborhoods and their preimages in \( \tilde{X} \). Note that if there were no rotations, the “fins” of \( \tilde{X} \) would get folded together, and this folding would not be a local homeomorphism.