1. (a) Prove that if \( p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0) \) is an \( n \)-sheeted covering space then \( p_* (\pi_1(\tilde{X}, \tilde{x}_0)) \) is a subgroup of \( \pi_1(X, x_0) \) of index \( n \).

(b) Find two 2-sheeted covering spaces of the torus \( S^1 \times S^1 \) that are not isomorphic to each other (as covering spaces). Can you find three?

2. (a) Find all covering spaces of the circle up to covering space isomorphism.

(b) Find all homomorphisms between these covering spaces.

(c) If you ignore the covering maps to \( S^1 \), what are the covering spaces of \( S^1 \) up to homeomorphism? That is, which spaces arise in part (a)?

3. (a,b,c) Same as question 2, but for covering spaces of the Möbius band. The Möbius band is the quotient of the square \( I \times I \) by the equivalence relation that identifies the point \((0, t)\) to the point \((1, 1-t)\) for each \( t \in I \).

Here is an alternate description of the Möbius band that may be useful. Let \( f: \mathbb{R} \times I \rightarrow \mathbb{R} \times I \) be the map \((x, t) \mapsto (x+1, 1-t)\). Then the Möbius band is the quotient of \( \mathbb{R} \times I \) by the equivalence relation whose equivalence classes are the orbits of \( f \). That is, \((x, t) \sim (x', t')\) if and only if \((x', t') = f^k(x, t)\) for some \( k \in \mathbb{Z} \).

4. Let \( X \) be the 3-fold dunce cap. That is, \( X \) is a 2-dimensional cell complex with one 0-cell, one 1-cell, and one 2-cell, whose attaching map \( S^1 \rightarrow X^1 \) winds 3 times around (note that \( X^1 \) is a circle). Describe a simply connected covering space of \( X \) and its covering map to \( X \). [Hint: it is a cell complex having 3 \( i \)-cells for \( i = 0, 1, 2 \).]