1(a) State the axioms for $B$ to be a basis.
(b) Define the topology $T$ generated by $B$.
(c) Show that if $B$ is a basis for a topology on $X$, then the topology generated by $B$ equals the intersection of all topologies on $X$ that contain $B$.

2(a) Recall the definition of $A + B$ for ordered sets $A$ and $B$: $a < b$ for all $a \in A$ and $b \in B$, and the orderings within $A$ and $B$ are unchanged. Each of the following sets is given the dictionary order. Identify the order type of each one as $\mathbb{Z}_+$ or $\mathbb{Z}_+ \times \mathbb{Z}_+$ or $\mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+$, etc.

(i) $\{0,1\} \times \{0,1\} \times \mathbb{Z}_+$
(ii) $\{0,1\} \times \mathbb{Z}_+ \times \{0,1\}$
(iii) $\mathbb{Z}_+ \times \{0,1\} \times \{0,1\}$

(b) Show that $\mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+ \times \cdots$ in the dictionary order is not well-ordered.

3. Let $X$ be an infinite set.
   (a) Show that there is an injective map $f: \mathbb{Z}_+ \to X$.
   (b) Show that for any $n \in \mathbb{Z}_+$, there is a bijection between $X$ and $X$ with $n$ points removed.

4. Let $X$ be a well-ordered set.
   (a) Define the least upper bound property and the greatest lower bound property.
   (b) Show that $X$ has the least upper bound property.
   (c) Show that $X$ has the greatest lower bound property.

5. Consider the two injective maps $\mathbb{Z}_+ \to \mathbb{Z}_+$ given by $f(x) = x + 2$ and $g(x) = x + 3$. The proof of the Schroeder–Bernstein Theorem constructs a bijection $\mathbb{Z}_+ \to \mathbb{Z}_+$ based on $f$ and $g$.
   (a) Describe the orbits of the construction in this example. How many are there? (It may help to draw a picture.)
   (b) Write down the bijection that the construction gives.

6(a) Show that $B = \{(a,b) \mid a < b, a \text{ and } b \text{ rational}\}$ is a basis for the standard topology on $\mathbb{R}$.
(b) Recall that the lower limit topology has basis $B_\ell = \{[a,b) \mid a < b\}$. Show that the basis $B' = \{[a,b) \mid a < b, a \text{ and } b \text{ rational}\}$ generates a topology different from the lower limit topology.