1. Let $V$ be the $\mathbb{R}$-vector space of polynomial functions $f : \mathbb{R} \to \mathbb{R}$ of degree at most 10, i.e. functions of the form
   \[ f(x) = a_{10}x^{10} + a_9x^9 + \cdots + a_1x + a_0, \quad a_i \in \mathbb{R} \]
   (a) Determine a basis of $V$ and the dimension of $V$
   (b) Consider the map $L : V \to \mathbb{R}$ given by
   \[ f \mapsto \int_0^1 f(x) \, dx. \]
   Show that $L$ is linear and determine the dimension of the kernel of $L$.

   (a) basis: \[ \{1, x, x^2, \ldots, x^{10}\} \] since every polynomial $f(x)$ as above is uniquely expressed as a linear combination of these terms. Dimension = 11.

   (b) \( (L1) \):
   \[ L(f(x) + g(x)) = \int_0^1 (f(x) + g(x)) \, dx = \int_0^1 f(x) \, dx + \int_0^1 g(x) \, dx = L(f(x)) + L(g(x)). \checkmark \]

   \( (L2) \):
   \[ L(cf(x)) = \int_0^1 (cf(x)) \, dx = c \int_0^1 f(x) \, dx = c \cdot L(f(x)). \checkmark \]
   So $L : V \to \mathbb{R}$ is linear.

   The image of $L$ is a subspace of $\mathbb{R}$, so must be $\{0\}$ or $\mathbb{R}$. It is $\mathbb{R}$, because any number can be $\int_0^1 f(x) \, dx$ for some $f \in V$ ($f(x) = c$, for ex.)
   So $\dim(\text{im} L) = 1$.

   Rank - Nullity \[ \Rightarrow \quad \dim(\ker L) = \dim V - \dim(\text{im} L) \]
   \[ = 11 - 1 \]
   \[ = 10. \]
2. Let $F: \mathbb{R}^2 \to \mathbb{R}^4$ be the linear map such that $f(0,1) = (4,3,7,3)$ and $f(1,1) = (6,7,7,7)$. Find the matrix for $F$, i.e. the matrix $A$ such that $F = L_A$.

A will be a $4 \times 2$ matrix, and its two columns are given by $f(1,0)$ and $f(0,1)$.

We have $f(0,1)$, but what is $f(1,0)$?

Write \((1,0) = (1,1) - (0,1)\)

then $f(1,0) = f(1,1) - f(0,1)$ by linearity

$$= (6,7,7,7) - (4,3,7,3)$$

$$= (2,4,0,4)$$.

So $A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \\ 0 & 7 \\ 4 & 3 \end{bmatrix}$. 
3. If $A = (a_{ij})$ is an $n \times n$ matrix, define the trace of $A$ to be $\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$. You may assume that this defines a linear map $\text{tr}: \text{Mat}_{n\times n}(K) \to K$.

(a) Show that $\text{tr}(AB) = \text{tr}(BA)$ for $A, B \in \text{Mat}_{n\times n}(K)$.

(b) If $B$ is invertible, show that $\text{tr}(B^{-1}AB) = \text{tr}(A)$.

(c) Prove that there are no matrices $A, B$ such that $AB - BA = I_n$.

(a) Say $A = (a_{ij}), B = (b_{jk})$, the $ij$-th entry of $AB$ is $\sum_{j=1}^{n} a_{ij} b_{ij}$.

So $\text{tr}(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} a_{ij}$.

By re-ordering the summation, the $ij$-th entry of $BA$ is $\sum_{j=1}^{n} a_{ji} b_{ij}$.

Therefore, $\text{tr}(BA) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} a_{ij}$.

(b) $\text{tr}(B^{-1}AB) = \text{tr}(B)(AB) = \text{tr}(A(BB^{-1})) = \text{tr}(AI) = \text{tr}(A)$.

(c) Since trace is linear,

$$\text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA).$$

But this is 0 by (a).

However, $\text{tr}(I_n) = n$.

Since $0 \neq n$, $AB - BA$ cannot be $I_n$. 
4. Recall that the rank of a linear map $L$ is the dimension of the image of $L$, and the nullity is the dimension of the kernel of $L$. Suppose $F: \mathbb{R}^3 \to \mathbb{R}^2$ and $G: \mathbb{R}^2 \to \mathbb{R}^3$ are linear maps.

(a) What are the possible ranks and nullities of $F$ and $G$?

(b) Prove that $G \circ F$ is not invertible.

(a) Since $\dim \mathbb{R}^3 = 3$, the possibilities for $F$ are:

- $\text{rk } F = 0$, $\text{null } F = 3$
- $\text{rk } F = 1$, $\text{null } F = 2$
- $\text{rk } F = 2$, $\text{null } F = 1$

Note: $\text{rk } F = 3$ is not possible because the image of $F$ is a subspace of $\mathbb{R}^2$.

The possibilities for $G$ are:

- $\text{rk } G = 0$, $\text{null } G = 2$
- $\text{rk } G = 1$, $\text{null } G = 1$
- $\text{rk } G = 2$, $\text{null } G = 0$

(b) Since $\text{rk } G \leq 2$, $G$ cannot be surjective (because $\dim \mathbb{R}^2 = 2$). Hence $G \circ F$ cannot be surjective, because $\text{Im}(G \circ F)$ is contained in $\text{Im}(G) = \mathbb{R}^3$. Hence $G \circ F$ is not bijective, and not invertible.

OR

Since $\text{nullity of } F \geq 3$, $F$ is not injective. Hence $G \circ F$ is not injective, because $\ker F(\neq 3)$ is contained in $\ker (G \circ F)$. Hence $G \circ F$ is not bijective. ...
5. True or False. Indicate your answer clearly on the left side. You do not need to show your work.

**T**  If \( F: V \to W \) is linear and surjective, and \( \dim V = \dim W \), then \( F \) is invertible.

\[ \ker F = \{0\} \] by rank-nullity.

**F**  If \( F: V \to W \) is linear then \( V \) is the direct sum of \( \ker F \) and \( \text{Im} F \).

\[ \text{Im} F \text{ is in } W, \text{ not } V \]

**F**  Every diagonal matrix is invertible. There may be zeros on the diagonal.

**T**  If \( U \cap W = \{0\} \) for subspaces \( U, W \subset V \) then \( \dim U + \dim W \leq \dim V \).

Choose bases for \( U, W \); then their union is linearly independent in \( V \).

**T**  Every linear map \( f: \mathbb{R} \to \mathbb{R} \) is of the form \( f(x) = ax \).

In the standard basis, a linear map \( \mathbb{R} \to \mathbb{R} \) has a \( 1 \times 1 \) matrix, \( (a) \).

The map is multiplication by \( a \).