1. (3) What does it mean for three functions $y_1(x), y_2(x), y_3(x)$ to be linearly dependent?

There are constants $A, B, C$ (not all zero) such that $Ay_1(x) + By_2(x) + Cy_3(x) = 0$.

2. (3,5) Suppose the functions $y_1(x) = x$, $y_2(x) = x^2$, and $y_3(x) = x^3$ are solutions to a linear homogeneous differential equation.

(a) Write down two more solutions to the differential equation.

$y_4(x) = x + x^2$

$y_5(x) = x^2 + x^3$ (any linear combination of $x, x^2, x^3$ is a solution)

(b) Use the Wronskian to determine whether $y_1, y_2, y_3$ are linearly independent.

$W(y_1, y_2, y_3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

$= x \begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} - x^2 \begin{vmatrix} 1 & 3x^2 \\ 0 & 6x \end{vmatrix} + x^3 \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix}$

$= 12x^3 - 6x^3 - 6x^3 + 2x^3$

$= 2x^3 \neq 0$

hence linearly independent.
3. (6) Solve the initial value problem \( y'' - 4y' + 3y = 0, \ y(0) = 7, \ y'(0) = 11. \)

Char. eq.: \( r^2 - 4r + 3 = 0 \)

\( (r-3)(r-1) = 0 \)

\( r = 3, 1 \)

\[ y = Ae^{3x} + Be^x \quad \rightarrow \quad 7 = Ae^0 + Be^0 \]

\[ y' = 3Ae^{3x} + Be^x \quad \rightarrow \quad 11 = 3A e^0 + Be^0 \]

\[ 7 = A + B \quad \rightarrow \quad 7 = 2 + B \]

\[ 11 = 3A + B \]

\( B = 5 \)

\( A = 2 \)

\[ y(x) = 2e^{3x} + 5e^x \]

4. (6) Convert the function \( x(t) = 5\cos(13t) - 12\sin(13t) \) into the form \( x(t) = C\cos(\omega t - \alpha) \). Use an exact expression (possibly involving \( \tan^{-1} \)) for \( \alpha \), rather than a decimal value.

\[
C = \sqrt{A^2 + B^2} = \sqrt{25 + 144} = 13
\]

\[ \omega = \frac{13}{2} \]

Fourth quadrant: \( \alpha = \tan^{-1}\left(\frac{B}{A}\right) + 2\pi \)

\[ x(t) = 13 \cos\left(13t - \left(\tan^{-1}\left(\frac{-12}{5}\right) + 2\pi\right)\right) \]
5. \((5,3)\) This problem concerns the differential equation \(y'' + 16y = e^{3x}\).

(a) Using the method of undetermined coefficients, find a particular solution to the equation.

\[
\begin{align*}
\text{Use } y_p &= Ae^{3x} \\
y_p' &= 3Ae^{3x} \\
y_p'' &= 9Ae^{3x}
\end{align*}
\]

\[
y'' + 16y = e^{3x} \quad \text{becomes} \quad 9Ae^{3x} + 16Ae^{3x} = e^{3x}
\]

\[
25A = 1 \\
A = \frac{1}{25}
\]

\[
y = \frac{1}{25}e^{3x}
\]

(b) Find a general solution to the equation.

\[
y_c = A\cos(4x) + B\sin(4x)
\]

\[
y = y_c + y_p = Ae^{3x} + \frac{1}{25}e^{3x}
\]

6. \((4,4)\) Consider the differential equation

\[
y^{(5)} - 6y^{(4)} + 21y^{(3)} - 62y'' + 108y' - 72y = x^2 e^{2x} + x \sin(3x).
\]

(a) Using the fact that \(r^5 - 6r^4 + 21r^3 - 62r^2 + 108r - 72 = (r - 2)^3 (r^2 + 9)\), find a general solution to the associated homogeneous equation. (That is, find the complementary solution \(y_c\).)

\[
r = 2, 2, 2, \pm 3i
\]

\[
y_c = Ae^{2x} + Bxe^{2x} + Cxe^{2x} + D\cos(3x) + E\sin(3x)
\]
(b) Using the table below, set up the appropriate form of a particular solution (but do not determine the values of the coefficients).

\[
\begin{align*}
P_m &= b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m \\
&= a \cos kx + b \sin kx \\
e^{\alpha}(a \cos kx + b \sin kx) \\
P_m(x)e^{\alpha} \\
P_m(x)(a \cos kx + b \sin kx)
\end{align*}
\]

\[
\begin{align*}
x'(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m) \\
x'(A \cos kx + B \sin kx) \\
x'(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)e^{\alpha} \\
x'(A \cos kx + B \sin kx) \\
x'(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)\cos kx + (B_0 + B_1 x + \cdots + B_m x^m)\sin kx
\end{align*}
\]

\[
Y_p = x^3(A_0 + A_1 x + A_2 x^2)e^{2x} + x\left[B_0 + B_1 x\right]\cos(3x) + x\left[C_0 + C_1 x\right]\sin(3x)
\]

7. (5,3) Recall the equation \(m x'' + c x' + kx = 0\) for free motion of a mass-spring-dashpot system.

(a) Find a general solution for the position function \(x(t)\) when \(m = 1\), \(c = 6\), \(k = 13\). Is this system underdamped or overdamped?

\[
x'' + 6x' + 13x = 0
\]

\[
r^2 + 6r + 13 = 0
\]

\[
r = \frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = -6 \pm 4i
\]

\[
r = -3 \pm 2i
\]

\[
x(t) = e^{-3t}\left[A \cos(2t) + B \sin(2t)\right]
\]

underdamped
(b) Suppose the damping constant $c$ is changed, and a solution to the new system has the graph shown below. Was $c$ increased or decreased? Explain briefly how you know.

The picture shows a critically damped or overdamped system.

Thus $c$ has increased.

8. (8) Consider the endpoint problem

$$y'' + 2y' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$ 

Is $\lambda = 2$ an eigenvalue? If so give an eigenfunction; otherwise say why not.

$$y'' + 2y' + 2y = 0 \quad \Rightarrow \quad r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$y(x) = e^{-x}(A \cos x + B \sin x)$$

$$y(0) = 0 \quad \Rightarrow \quad 0 = A \cos(0) + B \sin(0)$$

$$A = 0$$

So now $y = e^{-x}B \sin x$.

$$y(\pi) = 0 \quad \Rightarrow \quad 0 = e^{-\pi}B \sin(\pi)$$

true for any $B$.

So $y = B e^{-x}\sin(x)$ is a solution to the endpoint problem. $\lambda = 2$ is an eigenvalue. An eigenfunction is $y = B e^{-x}\sin(x)$ for any $B \neq 0$. 