1. (5) The following picture is a slope field for the differential equation $\frac{dy}{dx} = x + y$. Suppose $y(x)$ is a solution to the initial value problem $\frac{dy}{dx} = x + y$, $y(0) = 2$. Without finding $y$ explicitly, what can you say about $y(-250)$? What about $y(250)$?

In the negative $x$ direction, $y(x)$ approaches the line $y = -1 - x$. So $y(-250) \approx 249$.

In the positive $x$ direction, $y(x)$ gets very large, but we can’t estimate $y(250)$.

2. (9) This problem concerns the initial value problem $y' = 3y^{2/3}$, $y(2) = 0$.

(a) Use separation of variables to find a general solution to $y' = 3y^{2/3}$.

\[ \frac{dy}{dx} = 3y^{2/3} \]

\[ \int y^{-2/3} \, dy = \int 3 \, dx \]

\[ 3y^{1/3} = 3x + C \]

\[ y = \left( x + \frac{1}{3} C \right)^3 \]

(b) Use your general solution to solve the initial value problem $y' = 3y^{2/3}$, $y(2) = 0$.

\[ 0 = \left( 2 + \frac{1}{3} C \right)^3 \]

\[ 0 = 2 + \frac{1}{3} C \]

\[ \frac{1}{3} C = -2 \]

\[ y = (x - 2)^3 \]
(c) Find another solution to the initial value problem $y' = 3y^{2/3}$, $y(2) = 0$.

The singular solution

$$y = 0$$

(d) Can you find a third solution to the initial value problem $y' = 3y^{2/3}$, $y(2) = 0$?

\[ y(x) = \begin{cases} 
0 & x \leq 0 \\
(x-2)^3 & x > 0 
\end{cases} \]

3. (8) The differential equation \( \frac{dy}{dx} = (1 - y) \cos x \) is both separable and linear.

(a) Find a general solution using separation of variables.

\[
\int \frac{dy}{1-y} = \int \cos x \, dx \\
-\ln |1-y| = \sin x + C \\
\ln |1-y| = -\sin x - C
\]

\[
\begin{align*}
1-y &= Ce^{-\sin x} \\
y &= 1 - De^{-\sin x}
\end{align*}
\]

(b) Write the equation in linear form, find the integrating factor, and proceed to solve the equation, until you have one quantity equal to the integral of another quantity. (Then stop.)

\[
\frac{d}{dx} \left( e^{\sin x} y \right) = (\cos x) e^{\sin x}
\]

\[
\begin{align*}
p(x) &= \int \cos x \, dx \\
p(x) &= e^{\sin x}
\end{align*}
\]

\[
e^{\sin x} y = \int (\cos x) e^{\sin x} \, dx
\]
4. (8) Use an integrating factor to solve the initial value problem \( xy' + 3y = 3x^{-5/2}, \ y(1) = 3 \).

\[
y' + \frac{3}{x}y = 3x^{-5/2}
\]

\[
\mu = e^{\int \frac{3}{x} \, dx} = e^{3 \ln x} = x^3
\]

\[
x^3y' + x^3 \cdot \frac{3}{x} y = 3x^3 x^{-5/2}
\]

\[
\frac{d}{dx} (x^3 y) = 3x^{1/2}
\]

\[
x^3 y = \int 3x^{1/2} \, dx = 2x^{3/2} + C
\]

\[
y = 2x^{-3/2} + Cx^{-3}
\]

\[
y = 2x^{-3/2} + x^{-3}
\]

5. (5) Use the substitution \( p = \frac{dy}{dx} \) to transform the differential equation \( y'' = 2y(y')^3 \) into a first order differential equation involving only \( y, p, \) and \( \frac{dp}{dy} \). (Do not solve either equation.)

\[
p = \frac{dy}{dx}, \quad y'' = \frac{dp}{dx} = \frac{dp}{dy} \frac{dy}{dx} = \frac{dp}{dy} p
\]

\[
\frac{dp}{dy} p = 2y(p)^3
\]
6. (10) In a certain culture of bacteria, the number of bacteria increased by a factor of five in 10 hours. How long did it take for the population to double?

Note that the population $P(t)$ satisfies the differential equation $\frac{dp}{dt} = kP$. (You may leave your answer as an unsimplified, exact expression.)

\[
\frac{dp}{dt} = kP \quad \Rightarrow \quad \int \frac{dp}{p} = \int k dt
\]

\[
\ln |P| = kt + C
\]

\[
|P| = e^C e^{kt}
\]

\[
P(t) = De^{kt}
\]

Note $P(0) = D \cdot e^0 = D$, so $D$ is initial population.

We're told $P(10) = 5D$:

\[
5D = De^{k(10)} \quad \text{so} \quad e^{10k} = 5,
\]

\[
10k = \ln 5 \quad \Rightarrow \quad k = \frac{1}{10} \ln 5.
\]

Now put in $2D$ for $P$, and find $t$:

\[
2D = De^{\left(\frac{1}{10} \ln 5\right)t}
\]

\[
e^{\left(\frac{1}{10} \ln 5\right)t} = 2
\]

\[
\left(\frac{1}{10} \ln 5\right)t = \ln 2
\]

\[
t = 10 \frac{\ln 2}{\ln 5} \approx 4.31 \text{ hours}
\]
7. (10) Use the substitution $v = y^{-3}$ to find a general solution to $3y + x^3y^4 + 3xy' = 0$.

$$3xy' + 3y = -x^3y^4 \rightarrow y' + \frac{1}{x}y = \frac{-x^2}{3}y^4 \quad \text{Bernoulli, } n = 4.$$ 

$v = y^{-3}$, \quad $y = y^{-\frac{1}{3}}$, \quad $\frac{dv}{dx} = -\frac{1}{3}v^{-\frac{4}{3}} \frac{dv}{dx}$.

Substitute:

$$\frac{-1}{3}v^{-\frac{4}{3}} \frac{dv}{dx} + \frac{1}{x}v^{-\frac{1}{3}} = \frac{-x^2}{3} \left( v^{-\frac{1}{3}} \right)^4$$

Divide by $-\frac{1}{3}v^{-\frac{1}{3}}$:

$$\frac{dv}{dx} - \frac{3}{x}v = x^2, \quad \text{linear.} \quad \rho(x) = e^{\int \frac{-3}{x}dx} = e^{\frac{3}{x}} = x^{-3}.$$ 

$$\frac{d}{dx} \left( x^{-3}v \right) = x^2 \cdot x^{-3}$$

$$x^{-3}v = \int x^2dx = ln|x| + C.$$ 

$$v = x^3(\ln|x| + C).$$ 

$$y = \left[ x^3(\ln|x| + C) \right]^{-\frac{1}{3}} = x^1(\ln|x| + C)^{-\frac{1}{3}}$$

Bonus. (4) Find a solution to $x^2y'' + y = x^2 + 1$.

Try trial and error with polynomials. If you put in $y = Ax^2 + Bx + C$ (so $y'' = 2A$) you get

$$x^2(2A) + Ax^2 + Bx + C = x^2 + 1$$

So $3A = 1$, $B = 0$, $C = 1$.

$$y(x) = \frac{1}{3}x^2 + 1$$ is a solution.