1. (8 points) Consider the endpoint problem

\[ y'' + \lambda y = 0, \quad y(0) = 0, \quad y(2) = 0. \]

Is \( \lambda = 5 \) an eigenvalue? If so give an eigenfunction, and otherwise explain why not.

\[ y'' + 5y = 0 \quad r^2 + 5 = 0 \quad r = \pm \sqrt{5} \, i \]

\[ y = A \cos (\sqrt{5} \, x) + B \sin (\sqrt{5} \, x) \]

\[ y(0) = 0 : \quad 0 = A + 0 \quad \Rightarrow A = 0 \]

\[ y = B \sin (\sqrt{5} \, x) . \]

\[ y(2) = 0 : \]

\[ 0 = B \sin (2 \sqrt{5}) \]

\[ \neq 0 \]

So \( B = 0 \).

Hence the only solution is \( y = 0 \),
so \( \lambda = 5 \) is not an eigenvalue.
2. (8 points) Use the method of elimination to solve the following initial value problem:

\[ x' = -3x + 2y, \quad y' = -3x + 4y, \quad x(0) = 0, \quad y(0) = 2 \]

\( D: \begin{cases} \frac{(D+3)x}{-2y} = 0 \\ \frac{3x + (D-4)y}{0} = 0 \end{cases} \)

\( D-1: \begin{cases} (D+3)\dot{y} + 6y = 0 \\ y'' - y' - 6y = 0 \end{cases} \)

\[ r^2 - r - 6 = 0 \]
\[ (r - 3)(r + 2) = 0 \]
\[ r = 3, -2 \]

\[ y = Ae^{3t} + Be^{-2t} \]
\[ y(0) = 2 : \quad 2 = A + B \Rightarrow y = Ae^{3t} + (2-A)e^{-2t} \]

\[ y' = 3Ae^{3t} - 2(2-A)e^{-2t} \]

Using original equation:

\[ 3x = 4y - y' = 4Ae^{3t} + 4(2-A)e^{-2t} - 3Ae^{3t} + 2(2-A)e^{-2t} \]
\[ = Ae^{3t} + 6(2-A)e^{-2t} \]
\[ X = \frac{1}{3}Ae^{3t} + 2(2-A)e^{-2t} \]
\[ X(0) = 0 : \quad 0 = \frac{1}{3}A + 2(2-A) \Rightarrow \frac{5}{3}A = 4 \]
\[ \Rightarrow A = \frac{12}{5} \]

\[ x(t) = \frac{4}{5}e^{3t} - \frac{4}{5}e^{-2t} \]
\[ y(t) = \frac{12}{5}e^{3t} - \frac{2}{5}e^{-2t} \]
3. (12 points) Find the following Laplace transforms and inverse Laplace transforms:

(a) $L\{e^{\frac{t}{2}} \sin(3t)\}^2$

$L\{\sin(3t)\}^2 = \frac{3}{s^2 + 9}$, so $L\{e^{\frac{t}{2}} \sin(3t)\}^2 = \frac{3}{(s-\frac{1}{2})^2 + 9}$

(b) $L\{g(t)\}$, where $g'(t) = e^{\frac{t}{2}} \sin(3t)$

$L\{g'(t)\} = \frac{S}{(s-\frac{1}{2})^2 + 9} = S L\{g\} - g(0)$

$L\{g\} = \frac{1}{S} \left[ \frac{S}{(s-\frac{1}{2})^2 + 9} + g(0) \right]$

(c) $L^{-1}\{\frac{s}{(s-2)^2 + 4}\}$

$L^{-1}\{\frac{s}{(s-2)^2 + 4}\} = \frac{s-2}{(s-2)^2 + 4} + \frac{2}{(s-2)^2 + 4}$

$e^{2t} \cos(2t) + e^{2t} \sin(2t)$

(d) $L^{-1}\{\frac{3s+2}{(s^2+4)(s-1)}\}$

$\frac{3s+2}{(s^2+4)(s-1)} = \frac{As+B}{s^2+4} + \frac{C}{s-1}$

$3s+2 = (As+B)(s-1) + (s^2+4)C$

$s=1$: $5 = 5C \Rightarrow C = 1$.

$s=2i$: $6i+2 = (2A+Bi)(2i-1) + 0 = -4A + 2Bi - 2Ai - B$

$6i+2 = (2B-2A)i + (-4A-B)$

$6 = 2B-2A$, $2 = -4A -B \Rightarrow B = -4A - 2$

$6 = 2(-4A-2) - 2A \Rightarrow A = -1$, $B = 2$.

$L^{-1}\{F(s)\} = -\cos(2t) + \sin(2t) + e^t$
Use the Laplace transform to solve the following initial value problem:

\[ x'' - x' - 2x = 0, \quad x(0) = 0, \quad x'(0) = 2 \]

\[ \mathcal{L} \{x'(0)\} = X'(s) \quad \text{and} \quad \mathcal{L} \{x''(0)\} = s^2 X(s) - s(0) - 2 \]

So,

\[ \left( s^2 X(s) - 2 \right) - \left( s X(s) \right) - 2 X(s) = 0 \]

\[ X(s) = \frac{2}{s^2 - s - 2} = \frac{2}{(s-2)(s+1)} \]

\[ \frac{2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \]

\[ \begin{align*}
2 &= A(s+1) + B(s-2) \\
\text{at} \quad s = -1 &\quad 2 = -3B, \quad B = \frac{2}{3} \\
\text{at} \quad s = 2 &\quad 2 = 3A, \quad A = \frac{2}{3} 
\end{align*} \]

\[ X(s) = \frac{2}{3} \frac{1}{s-2} + \frac{2}{3} \frac{1}{s+1} \]

\[ x(t) = \frac{2}{3} e^{2t} - \frac{2}{3} e^{-t} \]
5. (8 points) Using the definition, find the Laplace transform of \( f(t) \), where \( f(t) = 0 \) for \( 0 \leq t \leq 2 \) and \( f(t) = 2 \) for \( t \geq 2 \).

\[
\mathcal{L}\{f(t)\} = \int_0^2 e^{-st} f(t) \, dt + \int_2^\infty e^{-st} f(t) \, dt
\]

\[
= \int_2^\infty e^{-st} \, dt = 2 \left[ \frac{-e^{-st}}{s} \right]_2^\infty
\]

\[
= \lim_{b \to \infty} \left( \frac{-2}{s} e^{-sb} - \frac{2}{s} e^{-2s} \right)
\]

\[
= \frac{2}{s} e^{-2s}
\]

6. (7 points) Transform the following system into a first order system:

\[
5x''' - 2x' + y \sin t = t^2 - y''
\]

\[
y' - x'' + 14e^t = 12x - y
\]

\[
\begin{align*}
X_1 &= x' \\
X_2 &= x_1' \\
Y_1 &= y' \\
5X_2' - 2X_1 + y \sin t &= t^2 - Y_1' \\
Y_1 - X_2 + 14e^t &= 12X - Y
\end{align*}
\]
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<td>$\frac{e^{-as}}{s}$ where $u(t-a)$ is 0 when $t &lt; a$, and 1 when $t \geq a$</td>
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<td>$\delta(t-a)$</td>
<td>$e^{-as}$ where $\delta(t-a)$ is a unit impulse at time $t = a$</td>
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