Homework Solutions for 4/23/2010

5.4 #8

$(3x + 2y)^7$ has terms $\binom{7}{j} (3x)^{7-j} (2y)^j$

and $x^8 y^9$ is the case $j=9$. So the coefficient is $\binom{7}{9} \cdot 3^8 \cdot 2^9 = \frac{17! \cdot 3^8 \cdot 2^9}{8! \cdot 9!} = 81,662,929,820$

5.4 #14

To make things easier, consider the cases $n$ even, $n$ odd separately.

**Even:** we want $1 = \binom{n}{0} < \binom{n}{1} < \binom{n}{2} < \cdots < \binom{n}{\frac{n}{2}}$

and $1 = \binom{n}{n} < \binom{n}{n-1} < \binom{n}{n-2} < \cdots < \binom{n}{\frac{n}{2}}$.

Note: Second statement follows from first since $\binom{n}{j} = \binom{n}{n-j}$ for all $j$. We also know by direct computation that $\binom{n}{0} = 1$. It remains to show that $\binom{n}{j-1} < \binom{n}{j}$ for $j < \frac{n}{2}$.

We have:

$$\frac{n!}{(j-1)!(n-j+1)!} < \frac{n!}{j!(n-j)!}$$

$$\Leftrightarrow \frac{j!}{(j-1)!} < \frac{(n-j+1)!}{(n-j)!}$$

(by cross-multiplying)

$$\Leftrightarrow j < n-j+1$$
The last statement is true because \( j \leq \frac{n}{2} \). Then \( \binom{n}{j-1} < \binom{n}{j} \) for such \( j \).

\[ \text{odd: writing } n = 2k + 1, \text{ we want:} \]
\[ 1 = \binom{n}{1} < \binom{n}{3} < \cdots < \binom{n}{k} \]
\[ \overset{!}{=} \]
\[ 1 = \binom{n}{1} < \binom{n}{2} < \cdots < \binom{n}{k+1} \]

Again, the second line follows from the first, and proving the first line is exactly the same as before.

**5.4 #20 Direct computation:**

Left side = \[ \frac{(n-1)!}{(k-1)!(n-k)!(k+1)!(n-k-1)!} \cdot \frac{n!}{k!(n-k+1)!} \]

Right side = \[ \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{n!}{(k-1)!(n-k+1)!(k+1)!(n-k)!} \]

Looking carefully, these are the same.
5.4 #24  \( c^n = \binom{p}{k} \), which is a positive integer. 
so \( n = \frac{p!}{k!(p-k)!} \) , hence \( N \cdot k! \cdot (p-k)! = p! \).

Recall the theorem: if \( p \mid abc \) and \( p \) is prime 
then \( p \mid a \) or \( p \mid b \) or \( p \mid c \). 
(Lehmer 2, p. 233)

Since \( p \mid p! \), we have \( p \mid (n)(k!)((p-k)!) \).
By the theorem, it suffices to show that 
\( p \nmid k! \) and \( p \nmid (p-k)! \) to conclude \( p \mid n \).

But \( k! \) and \( (p-k)! \) are products of numbers
that are all less than \( p \), so their
prime factors are all less than \( p \). Hence
\( p \) cannot divide \( k! \) or \( (p-k)! \).
So \( p \mid n \) (by the theorem quoted above).
Let $S = A \cup B$, $A, B$ disjoint sets each with $n$ elements.

Then \( \binom{2n}{2} \) = \# ways of choosing a subset of $S$ of size two.

Count three ways as follows. There are three separate (non-overlapping) cases:

- choose two from $A$
- choose two from $B$
- choose one from each.

The number of ways, in each case, is \( \binom{n}{2} \), \( \binom{n}{2} \), and \( n \cdot n \) respectively. By the sum rule, the total number is \( 2 \binom{n}{2} + n^2 \).

Hence \( \binom{2n}{2} = 2 \binom{n}{2} + n^2 \).

(b) Algebra: \( \binom{2n}{2} = \frac{(2n)!}{2!(2n-2)!} = 2n \frac{(2n-1)}{2} = 2n^2 - n \)

\[ 2 \binom{n}{2} + n^2 = 2 \frac{n!}{2(n-2)!} + n^2 = n(n-1) + n^2 = 2n^2 - n \]

So they are equal.
8.1 #4 (a) $a$ is taller than $b$
- not reflexive ($a$ is not taller than $a$)
- not symmetric
- anti-symmetric (arb $a$ and $b$ are never $a$)
- transitive

(b) $a$ and $b$ were born on the same day
- reflexive
- symmetric
- not anti-symmetric
- transitive

(c) $a$ has the same first name as $b$
- reflexive
- symmetric
- not anti-symmetric
- transitive

(d) $a$ and $b$ have a common grandparent
- reflexive
- symmetric
- not anti-symmetric
- not transitive

8.1 #7 (a) $x \neq y$
- not reflexive
- symmetric
- not anti-symmetric
- not transitive

(b) $xy > 1$
- not reflexive (take $x = y = 0$)
- symmetric
not anti-symmetric

transitive: since we are using integers, note that \( xy > 1 \implies x, y \neq 0 \) and \( x, y \) have the same sign.

Now it is easy to see that \( xRy \) and \( yRz \implies xRz \).

(c) \( \equiv \pm 1 \):

- not reflexive
- symmetric
- not antisymmetric
- not transitive

(d) \( x \equiv y \pmod{7} \):

- reflexive
- symmetric
- not antisymmetric
- transitive

(e) \( x \) is a multiple of \( y \):

- reflexive
- not symmetric
- not anti-symmetric (e.g. use 2, -2)
- transitive

(f) \( x \) and \( y \) both negative, or both nonnegative

- reflexive
- symmetric
- not antisymmetric
- transitive

(g) \( x = y^2 \):

- not reflexive
- not symmetric
- antisymmetric
- not transitive

\( G \) if \( x = y^2 \) and \( y = x^2 \) then \( x = x^4 \)

so \( x = 1 \), so \( y = 1 \). Hence \( x = y \).

(h) \( x > y^2 \):

- not reflexive
- not symmetric
- antisymmetric
- transitive

\( G \) "\( x > y^2 \) and \( y > x^2 \)" is never true
8.4 #8

(a) The relation "a = b" is symmetric and antisymmetric.

(b) Let $A = \{ a, b, c \}$ and let $R \subseteq A \times A$ be the relation $\{(a,b), (b,c), (c,b)\}$. Then $R$ is not symmetric (since $a R b$ and $b R a$) and is not antisymmetric ($b R c$ and $c R b$ and $b \neq c$).

8.4 #28

(a) $R_1 \cup R_2 = R_2$ since $R_1 \subset R_2$

(b) $R_1 \cap R_2 = R_1$ since $R_1 \subset R_2$

(c) $R_1 \oplus R_2 = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\}$

(d) $R_1 - R_1 = \text{same}$

8.4 #30

$S \circ R = \{(1,1), (1,2), (2,1), (2,2)\}$