1. Use induction to prove that $n! < n^n$ for every integer $n \geq 2$:
   What is the basis step?
   What is the inductive hypothesis?
   What do you need to prove in the inductive step?
   Complete these steps, and the proof.

   Let $P(n)$ be " $n! < n^n$".

   **Basis step:** $P(2)$, or " $2! < 2^2$"

   *This is true:* $2 < 4$.

   **Inductive hypothesis:** $P(k-1)$, or " $(k-1)! < (k-1)^{k-1}$"

   **Need to prove:** $P(k-1) \rightarrow P(k)$.
   We have: $(k-1)! < (k-1)^{k-1}$, want to prove $k! < k^k$.

   Write $k! = k \cdot (k-1)!$.

   Then $k \cdot (k-1)! < k \cdot (k-1)^{k-1}$ by ind. hyp.

   $< k \cdot k^{k-1}$ since $k-1 < k$

   $= k^k$.

   So $k! < k^k$, i.e. $P(k)$ is true.

   Since we have proved $P(2)$ and $P(k-1) \rightarrow P(k)$ for $k \geq 2$, by induction, we have proved $P(n)$ for all $n \geq 2$. 
2(a) Use the Euclidean Algorithm to find \( \gcd(9888, 6060) \).

\[
9888 = 1 \cdot 6060 + 3828 \\
6060 = 1 \cdot 3828 + 2232 \\
3828 = 1 \cdot 2232 + 1596 \\
2232 = 1 \cdot 1596 + 636 \\
1596 = 2 \cdot 636 + 324 \\
636 = 1 \cdot 324 + 312 \\
324 = 1 \cdot 312 + 12 \\
312 = 26 \cdot 12 + 0
\]

So \( \gcd(9888, 6060) = \gcd(312, 12) = 12 \).

2(b) Find the greatest common divisor and least common multiple of the numbers \( 3^75^37^3 \) and \( 2^{11}3^{5}5^3 \).

\[
\gcd(3^75^37^3, 2^{11}3^{5}5^3) = (\text{min of each exponent})
\]

\[
\text{lcm} = 2^{11}3^75^37^3
\]

2(c) Explain why \( ab = \gcd(a, b) \cdot \text{lcm}(a, b) \) for all positive integers \( a, b \).

If \( a = p_1^{m_1} \cdots p_k^{m_k} \), \( b = p_1^{n_1} \cdots p_k^{n_k} \) with each \( p_i \) prime and each \( m_i, n_i \geq 0 \), then

\[
\gcd(a, b) = p_1^{\min(m_1, n_1)} \cdots p_k^{\min(m_k, n_k)} \quad \text{and} \quad \text{lcm}(a, b) = p_1^{\max(m_1, n_1)} \cdots p_k^{\max(m_k, n_k)}
\]

and \( ab = p_1^{m_1+n_1} \cdots p_k^{m_k+n_k} \). The equation holds because \( \max(m_i, n_i) + \min(m_i, n_i) = m_i + n_i \) for each \( i \).
3(a) Give the definition for an infinite set \( S \) to be countable.

Then, suppose that \( S \) is countable and \( F \) is a finite set with \( n \) elements, disjoint from \( S \). Prove that \( F \cup S \) is countable.

Let \( F = \{a_0, a_1, \ldots, a_n\} \).

To show \( F \cup S \) is countable, we define a function

\[
F \cup S \xrightarrow{g} \mathbb{N}
\]

by:

\[
g(x) = \begin{cases} 
  i & \text{if } x = a_i \in F \\
  f(x) + n & \text{if } x \in S
\end{cases}
\]

(shift \( \mathcal{E} \)s of \( S \) forward by \( n \), map \( F \) to the freed-up numbers \( 0, \ldots, n-1 \)).

\( g \) is onto: every \( i \in \mathbb{N} \) is either \( g(a_i) \) (if \( i \leq n \)) or \( g(f^{-1}(i-n)) \).

\( g \) is \( 1 \)-1: \( g \) is clearly injective on \( F \), and is \( 1 \)-1 on \( S \) since if \( g(x) = g(y) \), then \( f(x) + n = f(y) + n \), so \( f(x) = f(y) \), hence \( x = y \) (since \( F \) is \( 1 \)-1). Finally, if \( a \in F \) and \( x \in S \), then \( g(a) \neq g(x) \), since \( g(a) \notin \mathbb{N} \) and \( g(x) \notin \mathbb{N} \).

3(b) What is the coefficient of \( x^9 \) in \((2-x)^{19}\)?

By the binomial theorem,

\[
(2-x)^{19} = \sum_{k=0}^{19} \binom{19}{k} (2)^{19-k} (-x)^k
\]

The \( x^9 \) term occurs when \( k=10 \), so this term is

\[
\binom{19}{10} 2^{10} (-x)^9 = \binom{19}{10} 2^{10} (-1)^9 x^9
\]

\[= \frac{-19! 2^{10}}{9! 10!} x^9 \]

This is the coefficient \( \frac{5}{10} \).
4. How many strings of six lowercase letters from the English alphabet contain the letter a? the letters a and b? the letters a and b in consecutive positions with a preceding b, with all the letters distinct?

(i) total number of strings = $26^6$
strings with no "a" : $25^6$
number of strings containing "a" : $26^6 - 25^6$

(ii) By inclusion-exclusion, the union of these sets has cardinality $25^6 + 25^6 - 24^6$.

so the number of strings containing "a" and "b" is $26^6 - (25^6 + 25^6 - 24^6)$.

(iii) there are 5 configurations: 

$ab_4$, $\overline{ab}_3$, $\overline{ab}_2$, $\overline{ab}_1$, $ab_0$.

In each configuration, you need 6 letters from a 4-permutation of $\{c, d, ..., z\}$. So there are $24 \cdot 23 \cdot 22 \cdot 21$ possibilities for each.

In all, $5 \cdot 24 \cdot 23 \cdot 22 \cdot 21$ strings.