1. Let \( f(x, y) = y^2 \cos(x) + 4(x^3 + 1)y \).

(a) At the point \((0, 2)\), find either the linearization \( L(x, y) \) or an equation of the tangent plane (your choice).

(b) Estimate \( f(0.02, 2.01) \). [A calculator should not be necessary.]

\[
\begin{align*}
(a) \quad f_x(x, y) &= -y^2 \sin x + 12x^2 y \\
f_y(x, y) &= 2y \cos x + 4(x^3 + 1) \\
f_x(0, 2) &= 0, \quad f_y(0, 2) = 4 + 4 = 8, \quad f(0, 2) = 4 + 8 = 12, \\
\quad L(x, y) &= 12 + 0(x-0) + 8(y-2) \\
\quad L(x, y) &= 8y - 4 \\
\text{or} \\
\text{tangent plane: } z &= 8y - 4
\end{align*}
\]

\[
\begin{align*}
(b) \quad f(0.02, 2.01) & \approx L(0.02, 2.01) \\
& = 8(2.01) - 4 \\
& = 16.08 - 4 \\
& = 12.08
\end{align*}
\]
2. Let \( f(x, y) = x^3 - 4xy. \)

(a) Find the directional derivative of \( f(x, y) \) at \((2, 1)\) in the direction of the origin.

(b) Find the maximum rate of change (among all directions) of \( f(x, y) \) at \((2, 1)\), and the direction in which it occurs.

\[
\nabla f (x, y) = 3x^2 - 4y, \quad \nabla f (x, y) = -4x

\]

\[
\nabla f = \langle 3x^2 - 4y, -4x \rangle

\]

\[
\nabla f (2, 1) = \langle 8, -8 \rangle

\]

Directional derivative = \( \nabla f (2, 1) \cdot u \)

\[
= \langle 8, -8 \rangle \cdot \langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \rangle

= \frac{-16}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \frac{-8}{\sqrt{5}}

\]

(b) max. rate of change

\[
| \nabla f (2, 1) |

= \sqrt{64 + 64} = \sqrt{128} = 8\sqrt{2}

\]

Direction = direction of \( \nabla f (2, 1) \)

\[
= \text{direction of} \langle 8, -8 \rangle

= \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle

\]
3. (a) For the function whose graph is shown below, determine the signs of the following partial derivatives: $f_x(0,3), f_y(0,3), f_{xx}(0,3), f_{yy}(0,3), f_{xy}(0,3)$.

\[
\begin{align*}
&f_x(0,3) = 0 \\
&f_y(0,3) < 0 \\
&f_{xx}(0,3) < 0 \\
&f_{yy}(0,3) > 0 \\
&f_{xy}(0,3) = 0
\end{align*}
\]

(\(f_x\) is 0 along y-axis, so \(y\) is unchanged as \(y\) is increased)

(b) Sketch the domain of the function \(f(x,y) = \sqrt{x^2+y^2-1} + \ln(y-x)\). Be sure to indicate which boundary points are in the domain by using dotted/solid lines, and open/closed dots at special points.

Domain of \(\sqrt{x^2+y^2-1}\) is \(x^2+y^2 \geq 1\) and \(y^2-x^2 \geq 0\), or \(x^2+y^2 > 1\).

Domain of \(\ln(y-x)\) is \(y-x > 0\), or \(y > x\).
4. (a) On the picture below, sketch the gradient vector $\nabla f(4, 3)$ for the function $f$ whose level curves are shown. Say how you chose the direction and length of this vector.

(b) Recall that a triangle with sides $a$ and $b$, with angle $\theta$ between them, has area $\frac{1}{2}ab\sin(\theta)$. Suppose that the length $a$ is increasing at a rate of 3 inches/sec and the length $b$ is decreasing at a rate of 2 inches/sec. How fast is the area of the triangle changing when $a = 40$ in, $b = 50$ in, and $\theta = \pi/6$?

\[
A(a, b, \theta) = \frac{1}{2}ab \sin(\theta), \quad a, b, \theta \text{ vary with } t
\]

\[
\frac{dA}{dt} = \frac{\partial A}{\partial a} \frac{da}{dt} + \frac{\partial A}{\partial b} \frac{db}{dt} + \frac{\partial A}{\partial \theta} \frac{d\theta}{dt} \quad \text{(chain rule)}
\]

\[
= \frac{b}{2} \sin(\theta) \frac{da}{dt} + \frac{a}{2} \sin(\theta) \frac{db}{dt} + \frac{ab \cos(\theta)}{2} \frac{d\theta}{dt}
\]

At this instant,

\[
\frac{dA}{dt} = \frac{50}{2} \sin\left(\frac{\pi}{6}\right) \cdot 3 + \frac{40}{2} \sin\left(\frac{\pi}{6}\right) \cdot (-2) + \frac{40 \cdot 50 \cos\left(\frac{\pi}{6}\right)}{2} \cdot 0
\]

\[
= \frac{150}{2} + \frac{-40}{2} + 0
\]

\[
= \frac{70}{4}
\]

\[\text{so area is increasing at a rate of } \frac{70}{4} \text{ inches}^2 \text{ per second.}\]