1. Let \( \mathbf{r}(t) = (\sin 3t, t, \cos 3t) \).
   (a) Find the tangent vector \( \mathbf{T}(t) \).
   (b) Find the curvature \( \kappa \) at the point \((0, \pi, -1)\).
   (c) Find the normal vector \( \mathbf{N}(t) \).

\[
\mathbf{r}'(t) = \langle 3 \cos 3t, 1, -3 \sin 3t \rangle
\]

\[
|\mathbf{r}'(t)| = \sqrt{9 \cos^2 3t + 1 + 9 \cos^2 3t} = \sqrt{10}
\]

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \langle \frac{3}{\sqrt{10}} \cos 3t, \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \sin 3t \rangle
\]

\[
\mathbf{T}'(t) = \langle -\frac{9}{\sqrt{10}} \sin 3t, 0, \frac{9}{\sqrt{10}} \cos 3t \rangle
\]

\[
|\mathbf{T}'(t)| = \sqrt{\frac{81}{10} \sin^2 3t + \frac{81}{10} \cos^2 3t} = \frac{9}{\sqrt{10}}
\]

\[
\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\frac{9}{\sqrt{10}}}{\sqrt{10}} = \frac{9}{10}
\]

\[
\kappa(\pi) = \frac{9}{10}
\]

\[
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1}{9} \langle -\frac{9}{\sqrt{10}} \sin 3t, 0, \frac{9}{\sqrt{10}} \cos 3t \rangle
\]

\[
= \langle -\sin 3t, 0, -\cos 3t \rangle
\]
2a. In an $xy$-coordinate system, sketch the level curves of the function whose graph is shown below.

2b. The following surfaces are graphs of a function $f$ and its partial derivatives $f_x$ and $f_y$. Determine which is which and label the graphs accordingly, with brief explanations.

Try looking at curves on the front and sides of the boxes.

For the front, the $f_y$ curve will be the derivative of the $f$ curve.

For the side, the $f_x$ curve will be the derivative of the $f$ curve.
3a. Sketch carefully the domain of \( f(x, y) = x \ln(xy) \sqrt{y - 4x} \).

Domain: \( xy > 0 \) and \( y - 4x > 0 \)

or \( x > 0 \) and \( y > 0 \)

\( xy > 0 \)

\( y - 4x > 0 \)

3b. Find the linearization of \( f(x, y) = x \sqrt{y} \) at the point \((1, 4)\). Then use it to estimate \( f(0.99, 4.003) \).

\[
\begin{align*}
  f(1, 4) &= 2 \\
  f_x(x, y) &= \sqrt{y} \\
  f_y(x, y) &= \frac{x}{2\sqrt{y}} \\
  f_x(1, 4) &= 2 \\
  f_y(1, 4) &= \frac{1}{4}
\end{align*}
\]

So

\[
L(x, y) = 2 + 2(x - 1) + \frac{1}{4}(y - 4)
\]

Then \( f(0.99, 4.003) \approx L(0.99, 4.003) \)

\[
\begin{align*}
  &= 2 + 2(-0.01) + \frac{1}{4}(0.003) \\
  &= 2 - 0.02 + 0.00075 \\
  &= 1.98075
\end{align*}
\]

(Actual value \( \approx 1.98074236 \ldots \))
4a. Find an equation of the tangent plane to the surface \( z = y^2 + z^2 - 2 \) at the point \((-1, 1, 0)\).

\[ 
\hat{u} + F(x, y, z) = -x + y^2 + 2z \] 

Thus, surface in \( F(x, y, z) = 2 \).

\[ \nabla F = \langle -1, 2y, 2z \rangle, \text{ so normal vector is } \nabla F(-1, 1, 0) = \langle -1, 2, 0 \rangle. \]

**Tangent Plane**:

\[
-(x+1) + 2(y-1) = 0
\]

4b. Five numbers \( x, y, z, u, \) and \( v \) are multiplied together. The first two are increasing at 5 units per second, and the last three are decreasing at 3 units per second. Find the rate of change of the product at a moment when each of the numbers is 10.

\[ f(x, y, z, u, v) = xyzuv. \]

\[ \frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} + \frac{df}{dz} \frac{dz}{dt} + \frac{df}{du} \frac{du}{dt} + \frac{df}{dv} \frac{dv}{dt} \]

\[ \uparrow \quad 5 \quad \quad \uparrow \quad 5 \quad \quad \uparrow \quad -3 \quad \quad \uparrow \quad -3 \quad \quad \uparrow \quad -3 \]

\[ \quad yzuv \\ \quad xzuv \\ \quad xyuv \\ \quad xyzv \\ \quad xyzu \]

When all numbers \( = 10 \), we have

\[ \frac{df}{dt} = 10^4 \cdot 5 + 10^y \cdot 5 + 10^y \cdot (-3) + 10^y \cdot (-3) + 10^y \cdot (-3) \]

\[ = [10000]. \]
5a. Show that the limit \( \lim_{(x,y) \to (0,0)} \frac{2x^2y}{x^3 + 3y^3} \) does not exist.

Try \( y = 0 \): \( f(x,0) = \frac{0}{x} \), zero on x-axis

Try \( y = x \): \( f(x,x) = \frac{2x^3}{4x^3} = \frac{1}{2} \) on line \( y = x \).

Since two paths to \((0,0)\) have different limiting values, the limit does not exist.

5b. Consider the functions \( f(x,y) = \frac{x^2 + 2}{1 + 2y^2} \) and \( g(x,y) = \frac{x^3 + 2x}{x + 2y^2} \).

(i) Where is \( f \) continuous? Where is \( g \) continuous?

(ii) Does \( \lim_{(x,y) \to (0,0)} g(x,y) \) exist? Explain carefully why or why not. [Hint: compare with \( f \).

i) \( f \) is continuous everywhere, since denominator is never zero.

\( g \) is continuous when \( x \neq 0 \), i.e. away from the y-axis.

ii) \( f \) and \( g \) agree on the domain of \( g \), so

\[
\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{(x,y) \to (0,0)} g(x,y)
\]

The former limit is \( f(0,0) \) since \( f \) is continuous.

So the limit exists, and equals \( f(0,0) = 2 \).
6a. Find the directional derivative of \( \frac{1}{xz} + \frac{1}{yz} \) at \((2, 2, 1)\) in the direction of the origin.

\[
\mathbf{u} = \frac{\langle -2, -2, -1 \rangle}{\sqrt{(-2)^2 + (-2)^2 + (-1)^2}} = \langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \rangle.
\]

\( \mathbf{u} \cdot \mathbf{F}(x, y, z) = \frac{1}{x z} + \frac{1}{y z} \), then \( \nabla \mathbf{F} = \langle \frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{x^2} + \frac{1}{y^2} \rangle \).

So \( \nabla \mathbf{F}(2, 2, 1) = \langle -\frac{1}{4}, -\frac{1}{4}, -1 \rangle \).

\[
\mathbf{D}_u \mathbf{F}(2, 2, 1) = \nabla \mathbf{F} \cdot \mathbf{u} = \left( -\frac{1}{4} \right)(-\frac{1}{3}) + \left( -\frac{1}{4} \right)(-\frac{1}{3}) + (-1)(-\frac{1}{3})
\]

\[
= \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = \frac{2}{3}
\]

6b. Find the maximum rate of change of \( x^2 y^3 z^4 \) at \((x, y, z) = (1, 1, 1)\), and the direction in which it occurs.

\( \mathbf{F}(x, y, z) = x^2 y^3 z^4 \), \( \nabla \mathbf{F} = \langle 2x y^3 z^4, 3x^2 y^3 z^4, 4x^2 y^3 z^3 \rangle \)

So \( \nabla \mathbf{F}(1, 1, 1) = \langle 2, 3, 4 \rangle \).

Max. rate of change = \( |\nabla \mathbf{F}| = \sqrt{4+9+16} = \sqrt{29} \).

Direction = \( \text{dir. of } \nabla \mathbf{F} = \frac{\langle 2, 3, 4 \rangle}{\langle 2, 3, 4 \rangle} = \langle \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \rangle \).

**Bonus** (+2 points) Here is a map showing a mountain stream and some contour lines. Which way is the stream flowing, and why?

The question is, which side is higher?

Left higher \( \Rightarrow \text{stream runs along the top of a ridge (not likely)} \)

Right higher \( \Rightarrow \text{stream runs along the bottom of a valley (likely)} \)

So, stream most likely flows to the left.