1. (8 points) Match the vector fields $F$ with the pictures I–IV:

(a) $F(x,y) = (y,x)$
(b) $F(x,y) = (2x, -2y)$
(c) $F(x,y) = (2x, 2y)$
(d) $F(x,y) = \left( \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$

2. (8 points) Are the following vector fields $F$ conservative? If so find a function $f$ with $\nabla f = F$, and otherwise give a reason why not.

(a) $F(x,y) = (3x \sin y, x^2 \cos y)$
\[ \frac{\partial P}{\partial y} = 3x \cos y, \quad \frac{\partial Q}{\partial x} = 2x \cos y \]
\[ \text{not equal, so } F \text{ is not conservative} \]

(b) $F(x,y) = (2x \cos y, -x^2 \sin y - 4)$
\[ \frac{\partial P}{\partial y} = -2x \sin y, \quad \frac{\partial Q}{\partial x} = -2x \sin y \]
\[ \text{Also } F \text{ is defined on } \mathbb{R}^2. \]
\[ F \text{ is conservative}, \quad f = x^2 \cos y + C(y) \]
\[ f = x^2 \cos y - 4y + D(x^2) \]
\[ \text{So } f(x,y) = x^2 \cos y - 4y + \tilde{E} \]
3. (8 points) The region $E$ is bounded by the surface $z = 1 - x^2$, the plane $y = 1 - x$, and the three coordinate planes. Express the integral $\iiint_E f(x, y, z) \, dV$ in three different ways, using $dV = dz \, dx \, dy$, $dV = dy \, dx \, dz$, and $dV = dy \, dz \, dx$. 

\[ \iiint_E f(x, y, z) \, dz \, dx \, dy \]

\[ \iiint_E f(x, y, z) \, dy \, dx \, dz \]

\[ \iiint_E f(x, y, z) \, dy \, dz \, dx \]
4. (8 points) Evaluate the following line integrals:

(a) \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F}(x, y) = (y^2, xy) \) and \( C \) is given by \( \mathbf{r}(t) = (t^2, t^3), 0 \leq t \leq 4 \).

\[
\int_0^4 \left< (t^3)^2, (t^2)(t^3) \right> \cdot \left< 2t, 3t^2 \right> \, dt
\]

\[
= \int_0^4 2t^7 + 3t^5 \, dt
= \frac{5}{8} t^8 \bigg|_0^4
= \frac{5}{8} 4^8
\]

(b) \( \int_C xy^2 \, ds \) where \( C \) is the upper half of the unit circle \( x^2 + y^2 = 1 \).

\[
\mathbf{r}(t) = \left< \cos t, \sin t \right>, \quad 0 \leq t \leq \pi
\]

\[
\mathbf{r}'(t) = \left< -\sin t, \cos t \right>
\]

\[
ds = \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt = dt
\]

\[
\int_C xy^2 \, ds = \int_0^\pi \cos t \sin^2 t \, dt
\]

\[ u = \sin t, \quad du = \cos t \, dt \]

\[
\int_0^\pi u^2 \, du = 0
\]
5. (8 points) Find the surface area of the part of the paraboloid \( z = x^2 + y^2 \) that lies below the plane \( z = 25 \). [Hint: draw a picture first. What is the region in the xy-plane?]

\[
x = f(x, y) = x^2 + y^2
\]

\[
x_x = 2x
\]

\[
x_y = 2y
\]

\[
D = \text{circle of radius } 5 \text{ in } x-y \text{ plane}
\]

\[
SA = \iint_D \sqrt{1 + (2x)^2 + (2y)^2} \, dA
\]

\[
\text{polar coords}
\]

\[
= \int_0^{2\pi} \int_0^5 \sqrt{1 + 4r^2 \cos^2 \theta + 4r^2 \sin^2 \theta} \, r \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^5 \sqrt{1 + 4r^2} \, r \, dr \, d\theta
\]

\[
\text{substitution: } u = 1 + 4r^2 \quad du = 8r \, dr
\]

\[
= \int_0^{101/4} \int_0^{101/2} \frac{u^{3/2}}{8} \, du \, d\theta
\]

\[
= \left. \frac{u^{5/2}}{10} \right|_1^{101/2} \, d\theta
\]

\[
= 2\pi \left( \frac{1}{10} \left( 101^{3/2} - 1 \right) \right)
\]

\[
= \frac{\pi}{6} \left( 101^{3/2} - 1 \right)
\]
6. (6 points) Let \( f(x, y, z) = \cos(x^2 + 2y - z) \). Let \( C_1 \) be the line segment from \((0, 0, 0)\) to \((1, 1, 0)\) and let \( C_2 \) be the curve on the surface \( z = e^{xy} \) lying directly above \( C_1 \). Find \( \int_{C_1} \nabla f \cdot dr \) and \( \int_{C_2} \nabla f \cdot dr \).

\[
\int_{C_1} \nabla f \cdot dr = f(1, 1, 0) - f(0, 0, 0) = \cos(3) - 1,
\]

\[
\int_{C_2} \nabla f \cdot dr = f(1, 1, e) - f(0, 0, 1) = \cos(3e) - \cos(-1).
\]

7. (8 points) Evaluate \( \iiint_E z \, dV \) where \( E \) is the region between the spheres \( x^2 + y^2 + z^2 = 1 \) and \( x^2 + y^2 + z^2 = 4 \) in the first octant. (Recall that the first octant means the region where \( x \geq 0, y \geq 0, \) and \( z \geq 0. \))

\[
\begin{align*}
\text{spherical coordinates:} & \quad \rho^2 = x^2 + y^2 + z^2 \\
& \quad 0 \leq \theta \leq \pi/2 \\
& \quad 0 \leq \varphi \leq \pi/2 \\
\end{align*}
\]

\[
\begin{align*}
\iiint_E z \, dV &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi \\
&= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \rho^2 \cos \varphi \sin \varphi \, d\rho \, d\varphi \\
&= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \rho^2 \cos \varphi \sin \varphi \, d\rho \\
&= \frac{\pi}{2} \int_0^1 u^2 \sin u \left( \frac{1}{4} \rho^4 \right) \, du \\
&= \frac{\pi}{2} \left( \frac{1}{4} \left[ u^4 \right]_0^1 \right) = \frac{\pi}{16},
\end{align*}
\]

\[
\frac{15\pi}{16}.
\]