1. (6 points) Evaluate $\iint_D (x+y) \, dA$ where $D$ is the region bounded by $y = 3x$ and $y = x^2$. [Sketch the region carefully first.]

\[
\int_0^3 \int_{x^2}^{3x} (x+y) \, dy \, dx = \int_0^3 \left( xy + \frac{1}{2} y^2 \right)_{x^2}^{3x} \, dx
\]

\[
= \int_0^3 3x^3 + \frac{9}{2} x^2 - x^3 - \frac{1}{2} x^4 \, dx
\]

\[
= x^3 + \frac{3}{2} x^3 - \frac{1}{4} x^4 - \frac{1}{10} x^5 \bigg|_0^3
\]

\[
= 27 + \frac{3}{2} (27) - \frac{1}{4} (81) - \frac{1}{10} (243)
\]

\[
= 22.95
\]

2. (6 points) Evaluate $\iint_R \cos(x^2 + y^2) \, dA$ by changing to polar coordinates. Here $R$ is the region that lies above the $x$-axis and within the circle $x^2 + y^2 = 16$.

\[
R: \quad 0 \leq r \leq 4, \quad 0 \leq \theta \leq \pi
\]

\[
\iint_R \cos(r^2) \, r \, dr \, d\theta
\]

\[
= \int_0^\pi \int_0^4 \cos(r^2) \, r \, dr \, d\theta
\]

\[
= \int_0^\pi \left( \frac{1}{2} \sin(r^2) \right)_{0}^{16} \, d\theta
\]

\[
= \int_0^\pi \frac{1}{2} \sin(16) \, d\theta
\]

\[
= \frac{\pi}{2} \sin(16)
\]
3a. (3 points) Sketch carefully the region for the integral \( \int_0^1 \int_{3y}^3 e^{x^2} \, dx \, dy \).

3b. (5 points) Change the order of integration and evaluate the integral.

\[
\int_0^3 \int_0^{3/3} e^{x^2} \, dy \, dx = \int_0^3 \left( e^{x^2} \bigg|_0^{3/3} \right) \, dx
\]

\[
= \int_0^3 \frac{1}{3} x e^{x^2} \, dx \quad \left[ du = 2x \, dx \right]
\]

\[
= \frac{1}{6} e^u \bigg|_0^q = \frac{1}{6} \left( e^q - 1 \right)
\]

4. (4 points) Write down a tree diagram and the chain rule for \( \frac{\partial w}{\partial q} \) when \( w = f(t, u, v) \), \( t = t(p, q, r, s) \), \( u = u(p, q, r, s) \), and \( v = v(p, q, r, s) \).

\[
\frac{\partial w}{\partial q} = \frac{\partial w}{\partial t} \frac{\partial t}{\partial q} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial q} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial q}
\]
5. (8 points) The length $l$, width $w$, and height $h$ of a box change with time. At a certain instant the dimensions are $l = 2$ m, $w = 1$ m, $h = 3$ m. Also $l$ and $w$ are increasing at a rate of 3 m/s and $h$ is decreasing at a rate of 4 m/s.

(a) Write a formula for the surface area $A$, and then write down the chain rule for $\frac{dA}{dt}$.

\[
A = 2lw + 2wh + 2lh
\]

\[
\frac{dA}{dt} = \frac{\partial A}{\partial l} \frac{dl}{dt} + \frac{\partial A}{\partial w} \frac{dw}{dt} + \frac{\partial A}{\partial h} \frac{dh}{dt}
\]

(b) At this instant find the rate at which the surface area is changing.

\[
\frac{dA}{dl} = 2w + 2h, \quad \frac{dA}{dw} = 2l + 2h, \quad \frac{dA}{dh} = 2l + 2w
\]

At this instant:

\[
\frac{dA}{dl} = 8 \text{ m}^2/\text{s}, \quad \frac{dA}{dw} = 10 \text{ m}^2/\text{s}, \quad \frac{dA}{dh} = 6 \text{ m}^2/\text{s}
\]

So

\[
\frac{dA}{dt} = (8)(3) + (10)(3) + (6)(-4) = 30 \text{ m}^2/\text{s}
\]
6. (8 points) A lamina occupies the region \( D \) bounded by \( y = 0 \), \( x = 1 \), and \( y = \sqrt{x} \), and has density \( \rho(x, y) = x \). Find the mass and the center of mass of the lamina.

\[
\text{mass} = \iint_D \rho(x, y) \, dA
\]

\[
= \int_0^1 \int_0^{\sqrt{x}} x \, dy \, dx = \int_0^1 x \left(\int_0^{\sqrt{x}} 1 \, dy\right) \, dx = \int_0^1 x \left(\sqrt{x} - 0\right) \, dx
\]

\[
= \frac{1}{2} \int_0^1 x^{3/2} \, dx = \frac{2}{5} \left[\frac{2}{5}\right] = \frac{2}{5}.
\]

\[
\bar{x} = \frac{1}{\frac{2}{5}} \int_0^1 x \left(\int_0^{\sqrt{x}} 1 \, dy\right) \, dx = \frac{5}{2} \int_0^1 x \left(\sqrt{x} - 0\right) \, dx
\]

\[
= \frac{5}{2} \left(\frac{2}{7} \cdot x^{7/2} \bigg| 1 \right) = \frac{5}{7}.
\]

\[
\bar{y} = \frac{1}{\frac{2}{5}} \int_0^1 y \left(\int_0^{\sqrt{x}} 1 \, dy\right) \, dx = \frac{5}{2} \int_0^1 x \left(\frac{1}{2} x^{2/2} \bigg| 0 \right) \, dx = \frac{5}{4} \int_0^1 x^2 \, dx
\]

\[
= \frac{5}{4} \left(\frac{1}{3} x^3 \bigg| 0 \right) = \frac{5}{12}.
\]

Center of mass = \((\bar{x}, \bar{y}) = \left(\frac{5}{7}, \frac{5}{12}\right)\).
7a. (4 points) A moth at \((1,2,2)\) wants to get warm as quickly as possible. The temperature is given by \(T(x,y,z) = \frac{x}{z} + 2x^2y\). In which direction should the moth fly?

\[
\nabla T = \left< \frac{1}{z} + 4xy, 2x^2, -\frac{x}{z^2} \right>
\]

\at \ (1,2,2) \ it \ is \ \left< \frac{1}{2} + 8, 2, -\frac{1}{4} \right>.

The moth should fly in the direction \(\left< 8\frac{1}{2}, 2, -\frac{1}{4} \right>\).

7b. (5 points) Use the gradient to find the directional derivative of \(f(x,y) = x^2\sin y - y^2\) at the point \((1, \frac{\pi}{2})\) in the direction of the origin. You do not need to simplify your answer.

\[
\nabla f = \left< 2x\sin y, x^2\cos y - 2y \right>
\]

\at \ (1, \frac{\pi}{2}) \ it \ is \ \left< 2, -\pi \right>. \ \text{Direction is}

\[
\hat{u} = \frac{\left< -1, -\frac{\pi}{2} \right>}{\left| \left< -1, -\frac{\pi}{2} \right> \right|} = \left< \frac{-1}{\sqrt{1 + \frac{\pi^2}{4}}}, \frac{-\frac{\pi}{2}}{\sqrt{1 + \frac{\pi^2}{4}}} \right>
\]

\[
D_\hat{u} f = \left< 2, -\pi \right> \cdot \hat{u} = \frac{-2}{\sqrt{1 + \frac{\pi^2}{4}}} + \frac{\pi^2/2}{\sqrt{1 + \frac{\pi^2}{4}}}
\]

7c. (5 points) The picture below shows level curves of a function \(f(x,y)\). Draw the gradient vectors at the indicated points. [Keep in mind the relative lengths of the vectors.]