1(a) Use Green's Theorem to evaluate \( \int_C xy^2 \, dx + 2x^2y \, dy \) where \( C \) is the triangle with vertices (0,0), (2,2), and (2,4), oriented positively.

Green's Theorem: \( \int_C P \, dx + Q \, dy = \iint_D (Q_x - P_y) \, dA \)

\[
P(x,y) = xy^2, \quad Q(x,y) = 2x^2y
\]

\[
\int_C xy^2 \, dx + 2x^2y \, dy = \iint_D (Q_x - P_y) \, dA
\]

\[
= \iint_D (4xy - 2xy) \, dA = \int_0^2 \int_0^2 2xy \, dy \, dx
\]

\[
= \int_0^2 \left[ \int_0^x y^2 \, dy \right] \, dx = \int_0^2 \left[ \frac{1}{3}x^3 \right] \, dx = \frac{1}{3} \left[ \frac{1}{2}x^4 \right]_0^2 = \frac{3}{4} (16 - 0) = 12
\]

1(b) Give a vector field \( \mathbf{F}(x,y) \) with the property that for any positively oriented simple closed curve \( C \), the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) measures the area enclosed by \( C \). Why does \( \mathbf{F} \) have this property?

Any \( \mathbf{F} \) with \( Q_x - P_y = 1 \) works, since by Green's Theorem,

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (Q_x - P_y) \, dA = \iint_D 1 \, dA = \text{Area} (D)
\]

\[
\mathbf{F}(x,y) = \left< 0, x \right> \quad \text{works}
\]

(also \( \left< -y, 0 \right> \), \( \frac{1}{2} \left< -y, x \right> \), etc.)
2(a) A certain function \( f(x, y) \) has gradient \( (e^{x^2}, ye^{y^2}) \). Its value at \((0, 0)\) is 5. Use a line integral to find \( f(0, 1) \). [Note: the function \( e^{x^2} \) cannot be integrated explicitly, so you will not be able to find a formula for \( f \).]

\[
\int_C \nabla f \cdot d\mathbf{r} = f(b) - f(a)
\]

\[
\int_0^1 (e^{x^2}, ye^{y^2}) \cdot (0, 1) \, dt = \int_0^1 te^{t^2} \, dt
\]

\[
= \frac{1}{2} \left. e^{t^2} \right|_0^1 = \frac{1}{2} (e - 1)
\]

So, \( \frac{1}{2} (e - 1) = f(0, 1) - 5 \)

\[
f(0, 1) = \frac{1}{2} (e - 1) + 5
\]

2(b) The figure below shows a vector field \( \mathbf{F}(x, y) = (P(x, y), Q(x, y)) \).

For each of the following quantities, say whether it is positive, negative, or zero:

- \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is the line segment from \((1, 0)\) to \((1, 2)\)
- \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is the line segment from \((0, 1)\) to \((2, 1)\)
- \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is the unit circle, oriented counter clockwise
- \( \frac{\partial P}{\partial x} \)
- \( \frac{\partial P}{\partial y} > 0 \)
- \( \frac{\partial Q}{\partial y} \)

Finally, is \( \mathbf{F} \) conservative? Say briefly why or why not.

\( \mathbf{F} \) is not conservative since integrating around a closed curve does not give zero.
3(a) A curve $C$ is parametrized as $r(t) = (\cos t, 2e^t)$ for $0 \leq t \leq \pi$. Express each of the following as ordinary integrals in $t$. Do not evaluate the integrals.

- $\int_C x^2y \, ds$
- $\int_C x^2y \, dx$
- $\int_C (x^2, y) \cdot dr$

\[ \frac{dr}{dt} = \left(-\sin t, 2e^t\right) \]
\[ ds = \sqrt{\sin^2 t + 4e^{2t}} \, dt \]
\[ dx = -\sin t \, dt \]

\[
\int_C x^2y \, ds = \int_0^\pi \cos^2 t \cdot 2e^t \sqrt{\sin^2 t + 4e^{2t}} \, dt
\]
\[
\int_C x^2y \, dx = \int_0^\pi \cos^2 t \cdot 2e^t (-\sin t) \, dt
\]
\[
\int_C (x^2, y) \cdot dr = \int_0^\pi -\cos^2 t \sin t + 4e^{2t} \, dt
\]

(b) Describe carefully in words the following surfaces (given with respect to spherical coordinates):

- $\rho = 3$  
  sphere of radius 3 centered at origin
- $\theta = \pi/2$  
  half-plane with boundary the $z$-axis and
- $\phi = \pi/2$  
  containing the positive $y$-axis
- the $xy$-plane
4. Let $E$ be the solid region that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the $xy$-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

(a) Draw the region $E$ and describe it using spherical coordinates.
(b) Find the volume of $E$.

\[
E = \{ (\rho, \theta, \phi) \mid 0 \leq \rho \leq 2, \quad 0 \leq \theta \leq 2\pi, \quad \frac{\pi}{4} \leq \phi \leq \frac{\pi}{2} \}
\]

\[
V_0(E) = \iiint_E d\nu
\]

\[
= \iiint_{E} \rho^2 \sin \phi \ d\rho \ d\theta \ d\phi
\]

\[
= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{0}^{2\pi} \int_{0}^{2} \rho^2 \sin \phi \ d\rho \ d\theta \ d\phi
\]

\[
= \frac{1}{3} \left. \rho^3 \right|_{0}^{2} \left( 2\pi \right) \left( -\cos \phi \right) \bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}}
\]

\[
= \left( \frac{8\sqrt{2}}{3} \right) \left( 2\pi \right) \left( \frac{\sqrt{2}}{2} \right) = \frac{8\sqrt{2}}{3} \pi
\]