1(a) [4] Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t), y = g(t)$. Indicate with arrows the direction in which the curve is traced as $t$ increases.

(b) [4] The figure shows the graph of $r$ as a function of $\theta$ in Cartesian coordinates. Use it to sketch the corresponding polar curve.
2. [6] Suppose \( \{a_n\} \) is a sequence which converges, to a limit \( L \).
(a) Explain in words why \( \lim_{n \to \infty} a_{n+1} = L \).

It's the same sequence, with the first term removed.

(b) Suppose this sequence is given by \( a_1 = 1 \), \( a_{n+1} = \frac{1}{1+a_n} \). Find \( L \). [Hint: use part (a).]

\[
L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{1}{1+a_n} = \frac{1}{1 + \lim_{n \to \infty} a_n} = \frac{1}{1 + L}
\]

So \( L^2 + L - 1 = 0 \)

\( L = \frac{-1 \pm \sqrt{5}}{2} \)

Since \( \frac{-1 + \sqrt{5}}{2} \) is positive,

\( L = \frac{-1 + \sqrt{5}}{2} \)

3. [9] For each sequence below, find the limit or say briefly why it doesn't converge:

(a) \( \sqrt{n} \cos \left( \frac{1}{n} \right) \)

\( \cos \left( \frac{1}{n} \right) \to 1 \) and \( \sqrt{n} \) increases without bound, so it \( \text{diverges} \)

(b) \( \frac{2^n}{5^{n+4}} = \frac{1}{5^4} \left( \frac{2}{3} \right)^n \)

The limit is 0.
(c) \((-1)^n \frac{n-3}{n+2}\) \(\frac{n-3}{n+2} \rightarrow 1\) but the sign alternates. It diverges.

4(a) [3] Sketch the infinite spiral \(r = e^{-\theta},\ 0 \leq \theta < \infty\).

(b) [5] Find the total length of this spiral (using an improper integral).

\[
\text{length} = \int_0^\infty ds = \int_0^\infty \sqrt{r^2 + \dot{r}^2} \, d\theta = \sqrt{2} \int_0^\infty e^{-\theta} \, d\theta
\]

\[
= \lim_{b \to \infty} \sqrt{2} \left[ -e^{-\theta} \right]_0^b
\]

\[
= \lim_{b \to \infty} \sqrt{2} \left[ -e^{-b} + e^0 \right] = \sqrt{2}
\]
5. [8] Find the surface area of the sphere of radius $R$, by rotating the appropriate portion of the parametric curve $x = R \cos t$, $y = R \sin t$, about the $x$-axis.

$$0 \leq \theta \leq \pi$$

$$x(t) = -R \sin t$$

$$y'(t) = R \cos t$$

$$ds = \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} \, dt$$

$$= R \, dt$$

$$SA = \int_0^\pi 2\pi y \, ds$$

$$= \int_0^\pi 2\pi R \sin t \, R \, dt$$

$$= 2\pi R^2 \int_0^\pi \sin t \, dt$$

$$= 2\pi R^2 \left[ -\cos t \right]_0^\pi$$

$$= 2\pi R^2 \left[ 1 + 1 \right]$$

$$= 4\pi R^2$$
6. [8] Find the area of the region that is inside the cardioid \( r = 4 + 4 \cos \theta \) and outside the circle \( r = 6 \) (draw a picture first!).

Intersection points:

\[
\begin{align*}
4 + 4 \cos \theta &= 6 \\
4 \cos \theta &= 2 \\
\cos \theta &= \frac{1}{2} \\
\theta &= \pm \frac{\pi}{3}
\end{align*}
\]

\[
\text{Area} = \int_{-\pi/3}^{\pi/3} \left[ \frac{1}{2} (4 + 4 \cos \theta)^2 - \frac{1}{2} (6)^2 \right] \, d\theta
\]

\[
= \frac{1}{2} \left[ \int_{-\pi/3}^{\pi/3} [16 + 32 \cos \theta + 16 \cos^2 \theta - 36] \, d\theta \right]
\]

\[
= \left[ (16 \cos \theta + 4 (1 + \cos 2\theta) - 10) \right]_{-\pi/3}^{\pi/3}
\]

\[
= \left[ 16 \sin \theta + 4 \theta + 2 \sin 2\theta - 10 \theta \right]_{-\pi/3}^{\pi/3}
\]

\[
= \frac{16 \sqrt{3}}{2} + \frac{4 \pi}{3} + 2 \frac{\sqrt{3}}{2} - 10 \frac{\pi}{3}
\]

\[
+ 16 \frac{\sqrt{3}}{2} + \frac{4 \pi}{3} + 2 \frac{\sqrt{3}}{2} - 10 \frac{\pi}{3}
\]

\[
= 18 \sqrt{3} - 4 \pi
\]
7(a) [4] True or False:
(i) If \( \{a_n\} \) and \( \{b_n\} \) converge then \( \{a_n + b_n\} \) converges.

true

(ii) If \( \{a_n\} \) and \( \{b_n\} \) are monotonic then \( \{a_n + b_n\} \) is monotonic.

false

7(b) [4] Give an example of:
(i) A bounded sequence that does not converge.
\[
\left\{ (-1)^n \right\} = (1, -1, 1, -1, ...)
\]

(ii) A convergent sequence that is not monotonic.
\[
\left\{ \frac{(-1)^n}{n} \right\} = (-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, ...) \to 0
\]

8. [5] Suppose a bee follows the trajectory \( x = 2 - 2\sin t \), \( y = t - \cos t \). \( 0 \leq t \leq 8 \). At what times is the bee flying horizontally?

\[
\frac{dx}{dt} = -2\cos t, \quad \frac{dy}{dt} = 1 + 3\sin t
\]

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + 3\sin t}{-2\cos t}
\]

flight is horizontal when \( 1 + 3\sin t = 0 \) (i.e. when \( \frac{dy}{dx} = 0 \))

i.e. \( \sin t = -\frac{1}{3} \)

i.e. \( t = \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}, ... \)

\[
\left[ t = \frac{3\pi}{2} \right] \text{ is the only one between } 0 \text{ and } 8.
\]

\[
\lim_{t \to \frac{3\pi}{2}} \frac{1 + \sin t}{-2 \cos t} = \lim_{t \to \frac{3\pi}{2}} \frac{\cos t}{-2\sin t} = -\frac{1}{2} \text{ by L'Hopital's rule.}
\]

\[
\text{Slope is indeed } 0 \text{ at } t = \frac{3\pi}{2}.
\]