1(a) (5 points) Find the absolute maximum and absolute minimum values of \( f(x) = x^4 - 2x^2 + 3 \) on the interval \([-2, 3]\).

\[ f'(x) = 4x^3 - 4x = 0, \quad 4x(x^2 - 1) = 0, \quad x = 0, \pm 1. \]

Critical points:

Check endpoints also:

\[ f(-2) = 11, \quad f(-1) = 2, \quad f(0) = 3, \quad f(1) = 2, \quad f(3) = 66. \]

Absolute max. is 66
Absolute min. is 2.

(b) (5 points) Sketch the graph of a function \( f \) that is continuous on \([1, 5]\) and has two local minima, one local maximum, no absolute maximum, and six critical numbers.

Diagram:

1. Local min
2. Vertical tangent
3. Local max
4. Corner (\( f' \) does not exist)
5. \( f' = 0 \)
6. Local min
2(a) (5 points) The figure shows the graphs of four functions. Out of the four, identify three of them that could represent the position, velocity, and acceleration of a car (and say which is which).

(b) (5 points) Find the equation of the tangent line to the ellipse $x^2 + xy + y^2 = 3$ at the point $(1,1)$. [Hint: use implicit differentiation.]

\[
2x + \left( x \frac{dy}{dx} + y \right) + 2y \frac{dy}{dx} = 0
\]

\[
\text{(diff. both sides w.r.t. } x)\]

\[
2x + y = (-2y - x) \frac{dy}{dx}
\]

\[
\frac{dy}{dx} = \frac{2x + y}{-2y - x}
\]

At $(1,1)$, slope is

\[
\frac{2(1) + 1}{-2(1) - 1} = -1
\]

Tangent line:

\[
(y - 1) = -1(x - 1)
\]
3. (10 points) Let \( f(x) = x^{2/3} \).

(a) Find the linearization of \( f(x) \) at the point \( a = 8 \).

\[
\begin{align*}
  \frac{d}{dx} f(x) &= \frac{2}{3} x^{-1/3} \\
  f'(8) &= \frac{2}{3} (8)^{-1/3} = \frac{1}{3} \\
  f(8) &= 4 \\
  \text{tangent line:} & \quad y - 4 = \frac{1}{3} (x - 8) \\
  y &= \frac{1}{3} x + \frac{4}{3}
\end{align*}
\]

\[
L(x) = \frac{1}{3} x + \frac{4}{3}
\]

(b) Estimate \((7.9993)^{2/3}\).

\[
\begin{align*}
  (7.9993)^{2/3} &\approx L(7.9993) \\
  &= \frac{7.9993}{3} + \frac{4}{3} = \frac{11.9993}{3} = 3.9997666...
\end{align*}
\]

Extra credit Using \( f''(x) \), can you say whether your estimate in (b) is an over- or under-estimate? Give an explanation with a picture.

\[
\begin{align*}
  \frac{d^2}{dx^2} f(x) &= -\frac{2}{9} x^{-4/3}, \\
  f''(8) &= -\frac{2}{9} (8)^{-4/3} < 0
\end{align*}
\]

Since \( f''(8) < 0 \), graph of \( y = x^{2/3} \) is concave down, so tangent line is above the graph.

So \( L(7.9993) \) is an overestimate.
4. (10 points)

(a) State the Mean Value Theorem.

If \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\) then there is a point \( c \) in \((a, b)\) where

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

(b) If \( f(0) = 4 \) and \( f'(x) \geq 6 \) for \( 0 \leq x \leq 10 \), what can you say about \( f(10) \)? Explain your (mathematical) reasoning.

Using MVT, with \( a = 0 \), \( b = 10 \). At some \( c \) between 0 and 10,

\[
f'(c) = \frac{f(10) - 4}{10 - 0}
\]

and \( f'(c) \geq 6 \).

So \( \frac{f(10) - 4}{10} \geq 6 \).

\[
\Rightarrow f(10) - 4 \geq 60 \Rightarrow f(10) \geq 64
\]

(c) Give a non-mathematical explanation using a driving analogy.

If you drive along a road for 10 hours and your speed never drops below 60 mph, then you will travel at least 600 miles.
5(a) (6 points) Use implicit differentiation to find $\frac{dy}{dx}$ for each of the curves $y = ax^3$ and $x^2 + 3y^2 = b$.

\[
\begin{align*}
\frac{dy}{dx} &= 3ax^2 \\
2x + 6y \frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{-2x}{6y}
\end{align*}
\]

(b) (4 points) Show that the curves $y = ax^3$ and $x^2 + 3y^2 = b$ are orthogonal. [Hint: Use $y = ax^3$ to eliminate $y$ from your expressions found in part (a).]

At any point $(x, y)$ the curve $y = ax^3$ has slope $3ax^2$ and the curve $x^2 + 3y^2 = b$ has slope

\[
\frac{-2x}{6y} = \frac{-2x}{6ax^3} = \frac{-1}{3ax^2}.
\]

The product of the two slopes is

\[
(3ax^2) \left(\frac{-1}{3ax^2}\right) = -1
\]

so the curves are orthogonal.
6. (10 points) A particle is moving along the curve \( y = \sqrt{x} \). As the particle passes through the point \((9, 3)\), its \( x \)-coordinate is increasing at a rate of 4 units per second. How fast is the distance from the particle to the origin changing at this instant?

Let \( D(t) \) = distance from origin. Then

\[ D(t)^2 = x(t)^2 + y(t)^2 \]

So

\[ 2D(t)D'(t) = 2x(t)x'(t) + 2y(t)y'(t) \]

At the given time we have

\[ x = 9, \quad y = 3, \quad D = \sqrt{90}, \quad x' = 4. \]

From \( y(t) = \sqrt{x(t)} \) we get

\[ y'(t) = \frac{1}{2} \frac{x'}{\sqrt{x(t)}} \]

So

\[ y' = \frac{1}{2} \frac{4}{3} = \frac{2}{3} \]

Putting everything in we get:

\[ 2 \sqrt{90} \cdot D'(t) = 2(9)(4) + 2(3) \left( \frac{2}{3} \right) \]

\[ D'(t) = \frac{38}{\sqrt{90}} \text{ units per second} \]