Self-Dual Representations and Signs

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$$\begin{array}{ccc}
\mathrm{Hom}_G(\pi, \pi^\vee) & & \mathcal{A} \\
\| & & \| \\
\left\{ \begin{array}{l} G \text{ maps between} \\ \pi \text{ and } \pi^\vee \end{array} \right\} & \simeq & \left\{ \begin{array}{l} \text{non-degenerate } G\text{-invariant} \\ \text{bilinear forms on } \pi \end{array} \right\} \\
f & \longrightarrow & b_f \\
f_b & \longleftarrow & b
\end{array}$$

Where

$$b_f(w_1, w_2) = \langle w_1, f(w_2) \rangle$$

$$\langle w_1, f_b(w_2) \rangle = b(w_1, w_2)$$

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Remark. We write $\boxed{c = \varepsilon(\pi)}$ and call it the sign of π.

Representations of some classical groups

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Remark

The above examples show that studying signs is not vacuous.

Main Objective

Objective

Determine the sign $\varepsilon(\pi)$ when π has non-zero vectors fixed under an Iwahori subgroup I in G (i.e when $\pi^I \neq 0$).

Note. For K a compact open subgroup of G, we define π^K as

$$\pi^K = \{w \in W \mid \pi(k)w = w, \forall k \in K\}.$$

Iwahori Subgroup

The Iwahori subgroup I is defined to be the inverse image of $B(k)$ under the map (reduction mod $\mathfrak{p}$) $G(\mathfrak{O}) \to G(k)$. ($k = \mathfrak{O}/\mathfrak{p}$ is the residue field)

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Example. Let $G = \mathrm{GL}(n, F)$ and B be the standard Borel subgroup (upper triangular matrices) in G. In this case the Iwahori subgroup is the collection of matrices of the following type.

$$I = \begin{pmatrix} \mathfrak{O}^\times & \mathfrak{O} & \cdots & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{O}^\times & \cdots & \mathfrak{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \cdots & \mathfrak{O}^\times \end{pmatrix}$$

A conjecture on signs

Conjecture

Let π be an irreducible smooth self-dual representation of G with non-zero vectors fixed under an Iwahori subgroup. Then $\varepsilon(\pi) = 1$.

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Remark

Most representations with an I fixed vector also have a K fixed vector. This lends some credence to the conjecture.

Main Result

Theorem

Let (π, W) be an irreducible smooth self-dual representation of G with non-zero vectors fixed under an Iwahori subgroup I. Suppose that π is also generic. Then $\varepsilon(\pi) = 1$.

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Remark

We assume π is generic in order to use Prasad's idea to study the sign $\varepsilon(\pi)$.

- Prasad shows that $\varepsilon(\pi) = \omega_\pi(s^2)$ for some special element $s \in T$ (T is a maximal split torus in G), whenever it exists.

Proving the Main Result

We consider two cases to prove the main result.

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Center of G is connected

- We prove the existence of the special element $s \in T$. In fact, we show that $s \in T(\mathfrak{O})$.
- Using Prasad's idea and the fact that $\pi^I \neq 0$, it follows that $\varepsilon(\pi) = 1$.

Proving the main result (contd $\cdots$)

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- We show that the lift $\tilde{\pi}$ can be chosen in such a way that it has non-zero vectors fixed under the Iwahori subgroup $\tilde{I}$ in $\tilde{G}$.
- We show that $\boxed{\varepsilon(\pi) = \varepsilon(\tilde{\pi})}$ and $\boxed{\varepsilon(\tilde{\pi}) = \omega_{\tilde{\pi}}(s^2)\chi(s)}$.
- Finally we show that the character χ is unramified to prove the main result.

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- Use structure of $\mathcal{H}$ (Iwahori-Matsumoto presentation or Bernstein presentation) to see if something can be said about the sign.

Thank You