A sample of Rota's mathematics

How can we define the real numbers $\mathbb{R}$, once we have defined the integers $\mathbb{Z}$?

Standard constructions, such as Dedekind cuts and equivalence classes of Cauchy sequences, are based on a two-step, geometric approach:

(1) Construct the rational numbers $\mathbb{Q}$.
(2) Fill in the "missing points of the line" to get $\mathbb{R}$.

There is nothing wrong with using geometric thinking (quite the contrary), but it is reasonable to ask whether there is a way to construct $\mathbb{R}$ from $\mathbb{Z}$ *without* using any geometric notions. Also, is it possible to avoid passing first to $\mathbb{Q}$?

The answers are "yes" and "yes." An elegant, purely algebraic construction that bypasses $\mathbb{Q}$ was given in a paper written by Rota and three other mathematicians:

> F. Faltin, N. Metropolis, B. Ross, G.-C. Rota, The real numbers as a wreath product, *Advances in Math.* 16 (1975), 278–304.

It is based on the natural idea of just regarding the real numbers as infinite decimals, but as we will see, there is a major difficulty to be surmounted.

What happens when we try to use the model of decimal numbers, or to make it simpler, base-2 numbers? A real number should be represented by a doubly-infinite string

$$\mathbb{A} = \cdots (a_{-n}) \cdots (a_{-2})(a_{-1}) a_0 \cdot a_1 a_2 a_3 \cdots$$

where

(1) For some N, $a_i = 0$ whenever $i < N$. That is, the string starts with infinitely many 0's.
(2) Each $a_i \in \{0, 1\}$, except that the first nonzero a_i might be -1.

So for example, we have

$$\cdots 0000101 \cdot 1100100000000000000000000 \cdots$$
$$\cdots 0000000 \cdot 0000101001100011000011 0 \cdots$$

These would represent the real numbers in the usual way. For example, the first one above, which has $1 = a_{-2} = a_0 = a_1 = a_2 = a_5$ and all other $a_i = 0$, would represent

$$(1 \times 2^2) + (1 \times 2^0) + (1 \times \frac{1}{2}) + (1 \times \frac{1}{2^2}) + (1 \times \frac{1}{2^5}) \ .$$

In general,

$$[\cdots (a_{-n}) \cdots (a_{-2})(a_{-1}) a_0 \cdot a_1 a_2 a_3 \cdots]$$

would represent the real number

$$\sum \frac{a_i}{2^i} \ .$$

Also, we declare

$$\cdots a_{k-1}a_k 01111111111111111111\cdots$$

to be equivalent to

$$\cdots a_{k-1}a_k 10000000000000000000\cdots$$

since these should represent the same real number.

The set of equivalence classes would indeed be $\mathbb{R}$, and sending

$$[\cdots(a_{-n})\cdots(a_{-2})(a_{-1})a_0 \cdot a_1 a_2 a_3 \cdots]$$

to

$$\sum \frac{a_i}{2^i}$$

would be a one-to-one correspondence with the real numbers produced by other constructions.

So what's the problem?

The problem: You must use your *new* description to *define* all the usual operations and other structures in $\mathbb{R}$, and *verify* their properties.

And when you define the operations, especially multiplication, you have to do a *lot* of carrying.

For example,

$$[\cdots 0a_0 \cdot a_1a_2a_3\cdots]\,[\cdots 0b_0 \cdot b_1b_2b_3\cdots] =$$
$$[\cdots 0(a_0b_0)\cdot(a_0b_1+a_1b_0)\,(a_0b_2+a_1b_1+a_2b_0)$$
$$(a_0b_3+a_1b_2+a_2b_1+a_3b_0)\cdots]$$

This product must now be simplified using carrying, perhaps requiring infinitely many carries.

But if you are allowed to do infinitely many carries, you can change any string to the zero string (and hence to any other string). For example:

$$\begin{aligned} 1\cdot 00000\cdots &\sim 0\cdot 200000\cdots \\ &\sim 0\cdot 040000\cdots \\ &\sim 0\cdot 008000\cdots \\ &\sim 0\cdot 000(16)00\cdots \\ &\sim 0\cdot 0000(32)0\cdots \\ &\sim \cdots \\ &\sim 0\cdot 000000\cdots \end{aligned}$$

How can we manage to do *enough* carrying, but not *too much*?

First, we use the time-honored technique of postponing the major difficulty, by working (for a while) with nonsimplified products:

Consider all infinite strings as above, but where a_i is allowed to be *any integer.* We still think in base 2, so for example

$$\cdots 003 \cdot 7(-4)000 \cdots$$

corresponds to the real number

$$(3 \times 2^0) + (7 \times \frac{1}{2}) + ((-4) \times \frac{1}{2^2}) = \text{“five-and-a-half”}$$

(Keep in mind that we haven't defined $\mathbb{R}$ yet, so this is only our secret intuition of what $\cdots 003 \cdot 7(-4)000 \cdots$ means.)

We say that $\cdots a_{-2}a_{-1}a_0 \cdot a_1a_2a_3 \cdots$ is *bounded* when $\sum \frac{a_i}{2^i}$ is absolutely convergent. This will ensure that it represents a real number.

Technical point: In the paper, all definitions and arguments are written using only integers, so that it is never necessary to introduce rational numbers. For example,

$$\sum \frac{a_i}{2^i} \text{ is absolutely convergent}$$

becomes

$$\text{there exists } N \text{ so that}$$
$$\text{for every } n, \; \sum_{i \leq n} |a_i| \, 2^{n-i} \leq 2^n N$$

What we need to do now is to define two bounded strings of integers to be "equivalent" in such a way that equivalent strings will correspond to the same real number.

Define $\mathbb{K}$ to be the infinite string $\cdots 0001{\cdot}(-2)0000$, which secretly represents $(1 \times 2^0) + ((-2) \times \frac{1}{2}) =$ "zero".

Notice that *adding* $\mathbb{K}$ to $\mathbb{A}$ corresponds to doing a carry at the 1's place:

$$\begin{aligned}(\cdots 0a_0 \cdot a_1a_2a_3\cdots) + (\cdots 01 \cdot (-2)0\cdots)\\ = \cdots 0(a_0+1)\cdot(a_1-2)a_2a_3\cdots\end{aligned}$$

Since

$$\mathbb{K}\,[\cdots 000 \cdot 1000\cdots] = \cdots 000 \cdot 1(-2)00\cdots \;\;,$$

adding $\mathbb{K}\,[\cdots 000 \cdot 1000\cdots]$ to a string has the effect of doing a carry in the halves place. In general, adding $\mathbb{K}\,\mathbb{C}$ to $\mathbb{B}$, for some integer string $\mathbb{C}$, has the effect of doing a bunch of carries to $\mathbb{B}$.

Now, define $\mathbb{A}$ to be equivalent to $\mathbb{B}$ when there exists a *carry string* $\mathbb{C}$ so that

$$\mathbb{A} = \mathbb{B} + \mathbb{K}\,\mathbb{C}$$

where $\mathbb{C} = \cdots c_{-1}c_0 \cdot c_1c_2\cdots$ is a *carry string* when

(1) $\mathbb{K}\,\mathbb{C}$ is bounded, and
(2) $\displaystyle\lim_{n\to\infty}\frac{c_n}{2^n} = 0.$

A couple of examples should convince you that this definition of equivalence is at least a reasonable attempt to allow the right amount of carrying.

First,

$$\cdots 0001 \cdot 0000 \cdots \sim \cdots 0000 \cdot 1111\ ,$$

since

$$(\cdots 0001 \cdot 0000 \cdots) =$$
$$(\cdots 0000 \cdot 1111) + \mathbb{K}\,(\cdots 0001 \cdot 1111 \cdots)$$

with $\cdots 0001 \cdot 1111 \cdots$ a carry string since $\lim \frac{c_n}{2^n} = \lim \frac{1}{2^n} = 0.$

On the other hand,

$$\cdots 0001 \cdot 0000 \cdots \not\sim \cdots 0000 \cdot 0000\ .$$

We *do* have

$$(\cdots 0001 \cdot 0000 \cdots) =$$
$$(\cdots 0000 \cdot 0000) + \mathbb{K}\,(\cdots 0001 \cdot 248(16)(32) \cdots)$$

but $\cdots 0001 \cdot 248(16)(32) \cdots$ is not a carry string since $\lim \frac{c_n}{2^n} = \lim \frac{2^n}{2^n} = 1 \neq 0.$

Confirmation that this is the correct amount of carrying to allow comes when the authors prove the following theorem:

Theorem. *Every bounded string is equivalent to a unique "clear string," i. e. a string in which:*

(1) *The first nonzero digit is* 1 *or* -1.
(2) *All later digits are either* 1 *or* 0.
(3) *The string does not end in all* 1*'s*
($0111\cdots \sim 1000\cdots$ *)*
(4) *If the first digit is* -1*, then the next one is* 0
($(-1)1 \sim 0(-1)$ *)*

So the equivalence classes do correspond to the base-2 decimals that we originally wanted to use in our definition.

Rota and his coauthors check that the operations

$$\mathbb{A} + \mathbb{B} = \mathbb{C} \text{ where } c_i = a_i + b_i$$

and

$$\mathbb{A}\,\mathbb{B} = \mathbb{C} \text{ where } c_i = \sum a_n b_{i-n}$$

are well-defined and have all the usual properties, and that sending $\cdots a_{-1}a_0 \cdot a_1 a_2 \cdots$ to $\sum a_i 2^{-i}$ really does define an isomorphism of fields to the real numbers as they are traditionally defined. The new definition works!

(A tricky point when defining division is that the multiplicative inverse of a bounded string need not be bounded. But the multiplicative inverse of a *clear* string is bounded, so can be used to define the multiplicative inverse.)

So what have we learned?

The real numbers *can* be constructed without invoking geometric thinking, and without first constructing the rational numbers. We can use a decimal-type representation, but we have to be careful to allow just the right amount of carrying.

This may or may not be the *best* way to think of the real numbers in most contexts, but it gives us a deeper understanding of the real numbers and their relation to $\mathbb{Z}$ and $\mathbb{Q}$.

This is very much in keeping with Rota's thinking that mathematics is not just a quest to solve problems, it is also a quest to *understand* the mathematical universe as clearly and as deeply as possible.

For algebraists: How can you construct the p-adics?

Answer: Take $\mathbb{K} = \cdots 000p \cdot (-1)000 \cdots$.