What is f'?

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"There are many roads to Nirvanah."

Yours may be different from mine.

f' can be defined in many ways.

My least favorite definition is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \ .$$

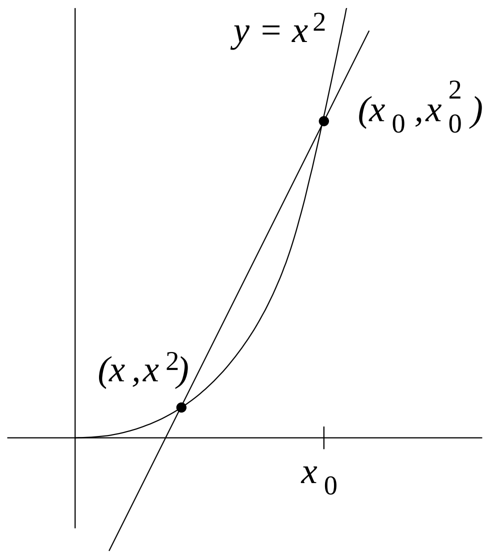
Although this happens to be an algebraically convenient way to verify some of the basic formulas, I find it to be one of the least useful ways to think about the derivative.

The underlying idea of f' is *linear approximation,* so I try to focus on this idea, rather than tacking it on as an afterthought.

To introduce the derivative, I like to use a two-step approach. I make no claim to originality, on the contrary I believe that this approach is "retro."

Step 1: Calculate a nontrivial derivative

This can be done, and the geometric meaning of the derivative can be explored, *before* introducing limits:



Consider all the lines through (x_0, x_0^2).
The line of slope m has equation $y - x_0^2 = m(x - x_0)$, so its intersections with the graph of $y = x^2$ are exactly the solutions of:

$$x^2 - x_0^2 = m(x - x_0)$$
$$x^2 - mx + (mx_0 - x_0^2) = 0 \ .$$

The discriminant of this quadratic is

$$m^2 - 4mx_0 + 4x_0^2 = (m - 2x_0)^2 \ ,$$

so there are two distinct intersection points except when the slope satisfies $m = 2x_0$.

By the way, this generalizes easily to compute the derivative of x^n without use of limits:

The intersections of the line of slope m through (x_0, x_0^n) and the graph of $y = x^n$ are the solutions of

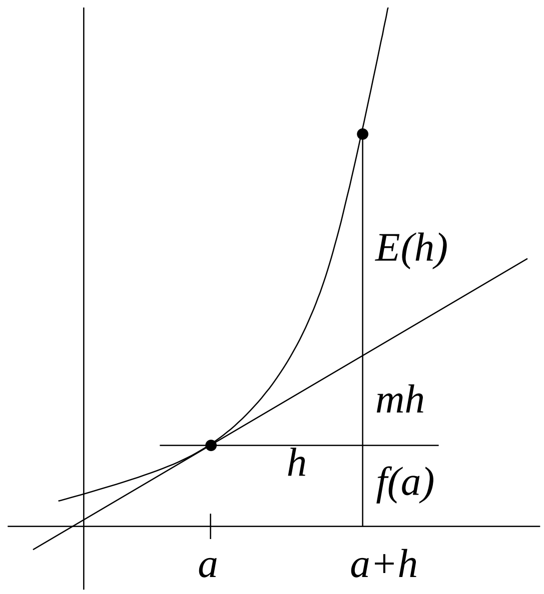
$$x^n - x_0^n = m(x - x_0)$$
$$(x - x_0)(x^{n-1} + x^{n-2}x_0 + \cdots + x_0^{n-1} - m) = 0$$

Consider what happens when you vary m. When the line becomes tangent, two roots coalesce into one, so x_0 appears twice as a root of the polynomial. So x_0 gives 0 in the factor

$$x^{n-1} + x^{n-2}x_0 + \cdots + xx_0^{n-2} + x_0^{n-1} - m \ ,$$

which says exactly that $m = nx_0^{n-1}$.

Step 2: After the basic idea of limits becomes available, define $f'(a)$ using the best linear approximation.



Define $f'(a)$ to be the choice of m (if one exists) such that $E(h)$ defined by

$$f(a+h) = f(a) + mh + E(h)$$

satisfies

$$\lim_{h \to 0} \frac{E(h)}{h} = 0 \ .$$

This puts the focus on linear approximation, in particular on the *error of linear approximation*, rather than on limits.

And this is the definition that generalizes trivially to functions from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Another advantage is that the definition

$$f(a+h) = f(a) + f'(a)h + E(h)$$

contains the basic idea of Taylor's formula.

When you get to the Mean Value Theorem, and want to show a use for it other than the contrived examples given in the book, you can apply it twice to give an upper bound for the error $E(h)$:

$$\begin{gathered} E(h) = f(a+h) - f(a) - f'(a)h \\ = f'(c)h - f'(a)h = f''(c_1)\,(c-a)\,h \end{gathered}$$

so

$$|E(h)| \leq Mh^2$$

where M is the maximum of $|f''(x)|$ between a and $a+h$.

And as soon as you get to integration by parts, you can use it to give a precise formula for the error. Calculate that

$$\int_0^h (h-t) f''(a+t)\,dt$$
$$= (h-t)f'(a+t)\Big|_0^h + \int_0^h f'(a+t)\,dt$$
$$= -f'(a)\,h + f(a+h) - f(a) = E(h)$$

so if m and M are the minimum and maximum values of $f''(x)$ between a and $a+h$, we have

$$\int_0^h (h-t)\,m\;dt \le E(h) \le \int_0^h (h-t)\,M\;dt$$
$$m\,\frac{h^2}{2} \le E(h) \le M\,\frac{h^2}{2}$$

and the Intermediate Value Theorem tells us that

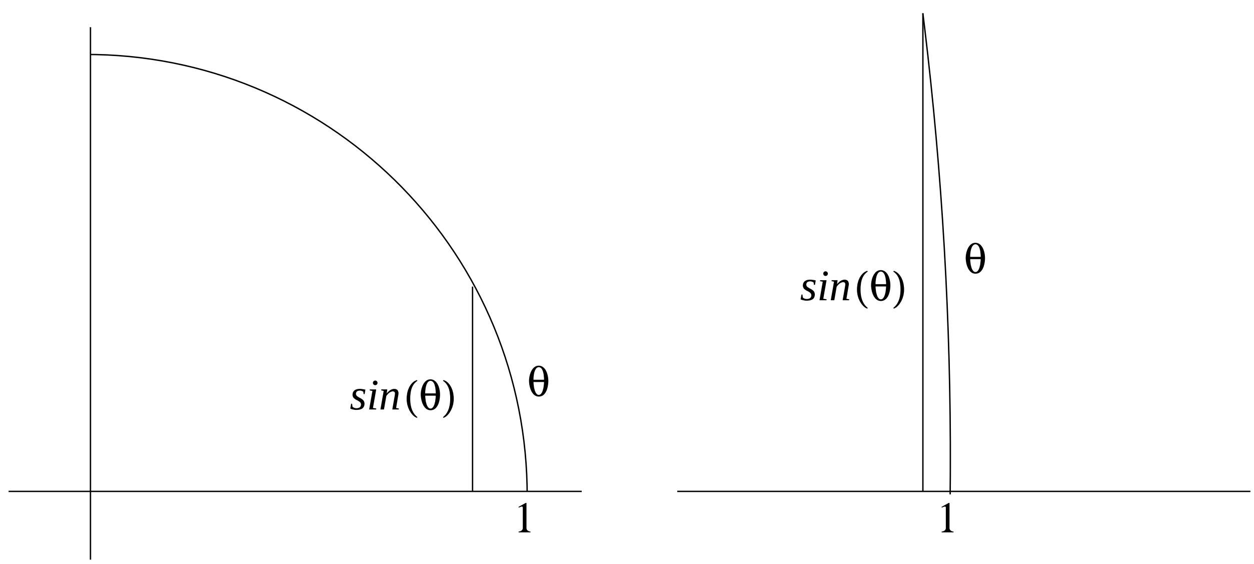
$$E(h) = f''(c)\,\frac{h^2}{2}$$

for some c between a and $a+h$.

I try to be conscious of the need to convey the underlying geometric reason why something works, rather than give formal, algebraic arguments that indulge my personal need for "proof."

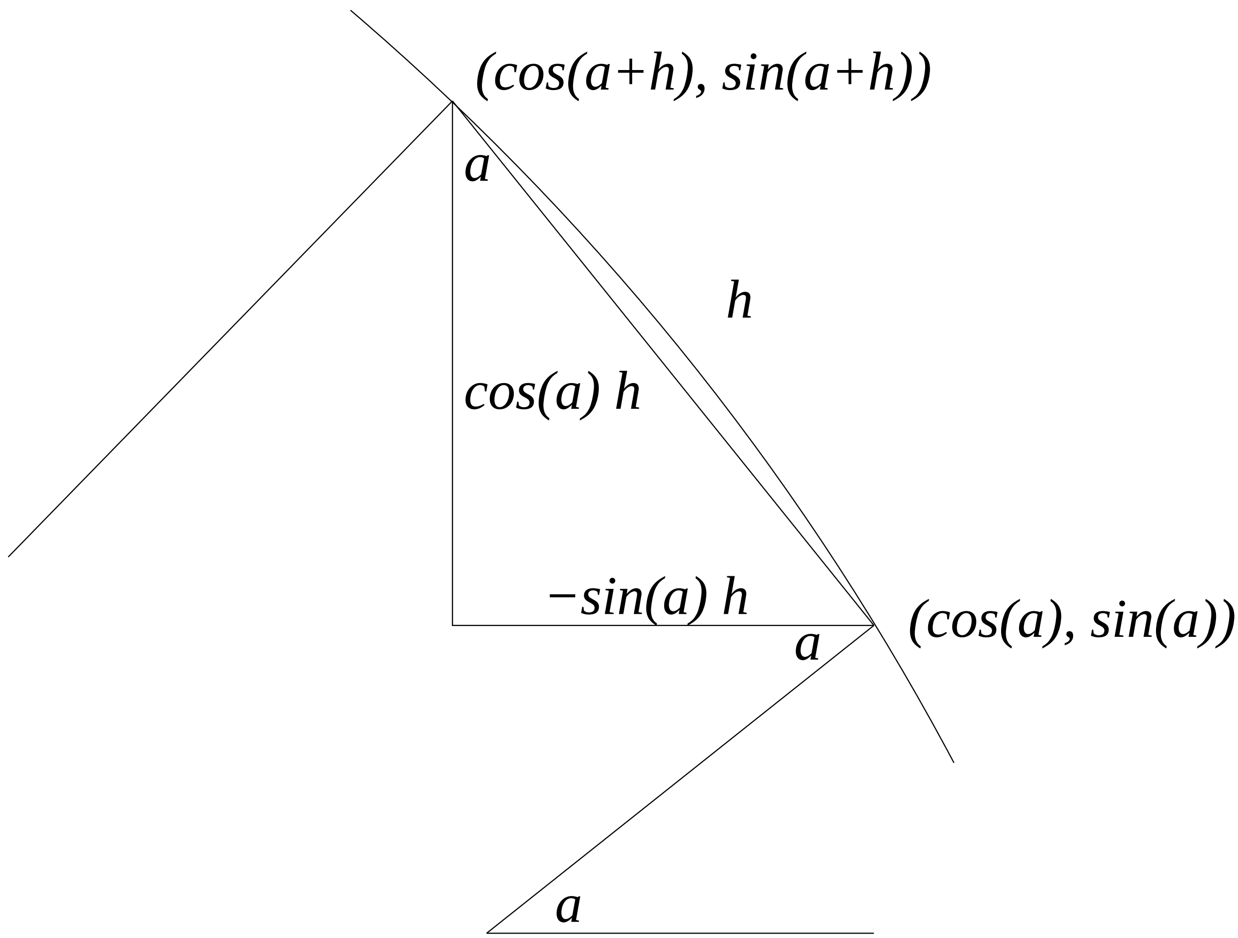
Here are some other examples of this teaching philosophy.

$$\text{Why is } \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1?$$



which also shows that $\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0$.

Why is $\dfrac{d}{d\theta}(\sin(\theta)) = \cos(\theta)$?



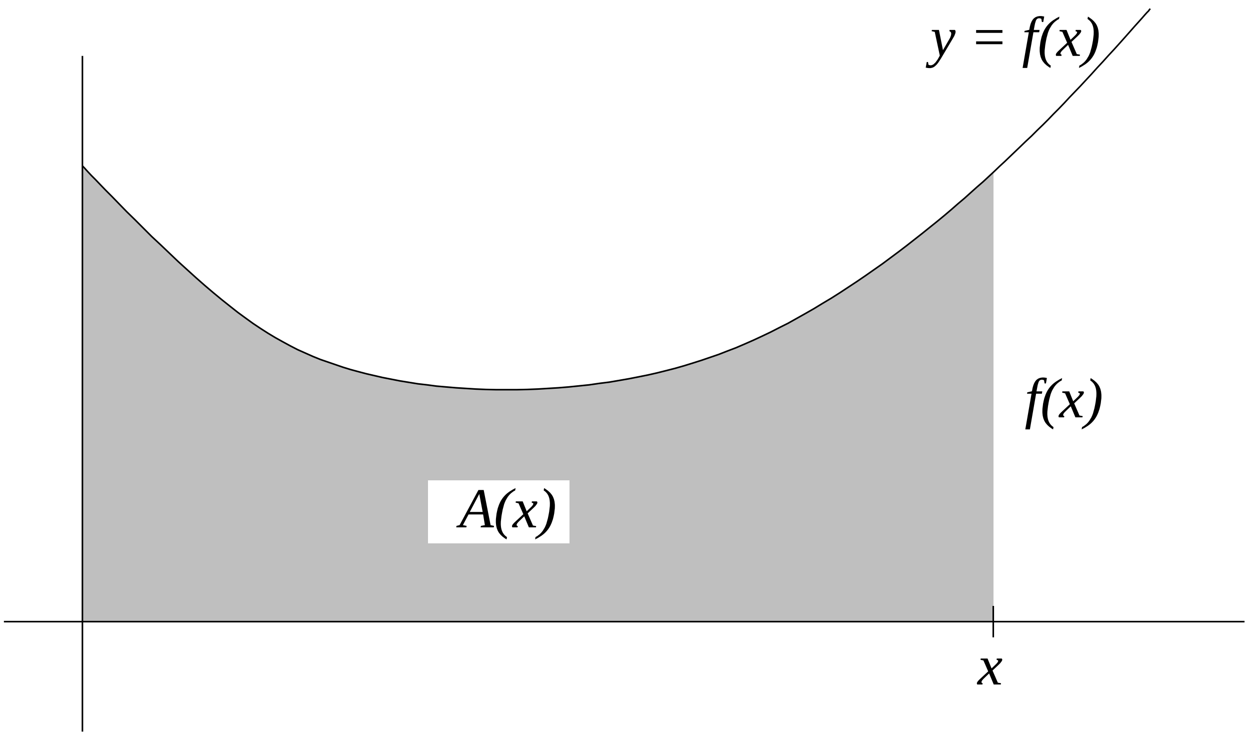
If you *do* want to prove algebraically that $\frac{d}{d\theta}\sin(\theta) = \cos(\theta)$, write

$$\sin(a+h) = \sin(a) + \cos(a)h$$
$$+(\cos(h)-1)\sin(a) + (\sin(h)-h)\cos(a)$$

and observe that

$$\lim_{h\to 0} \frac{\cos(h)-1}{h}\ \sin(a) + \Big(\frac{\sin(h)}{h} - 1\Big)\ \cos(a) = 0\ .$$

Why is the Fundamental Theorem of Calculus true?

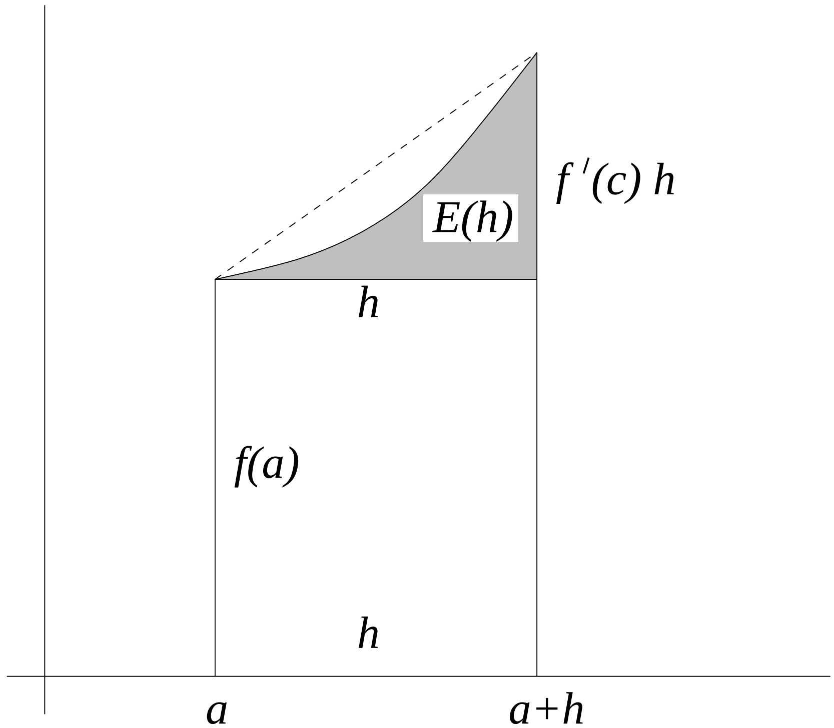


As one changes x, the rate at which $A(x)$ is increasing is proportional to $f(x)$.

That is, $A'(x) = kf(x)$ for some constant k.

Checking one example (such as $y = 2x$, for which $A(x) = x^2$ by the formula for the area of a triangle) shows that $k = 1$.

Why is the Fundamental Theorem of Calculus true?



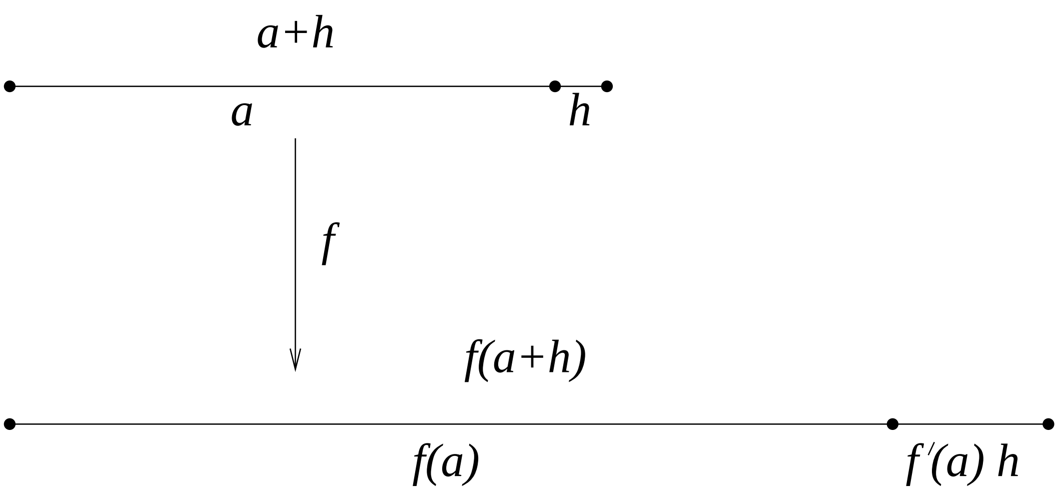
From this diagram, $A(a+h) = A(a) + f(a)h + E(h)$, where $E(h)$ is approximately the area of a triangle whose area is $\frac{1}{2}f'(c)h^2$ for some c between a and $a+h$.

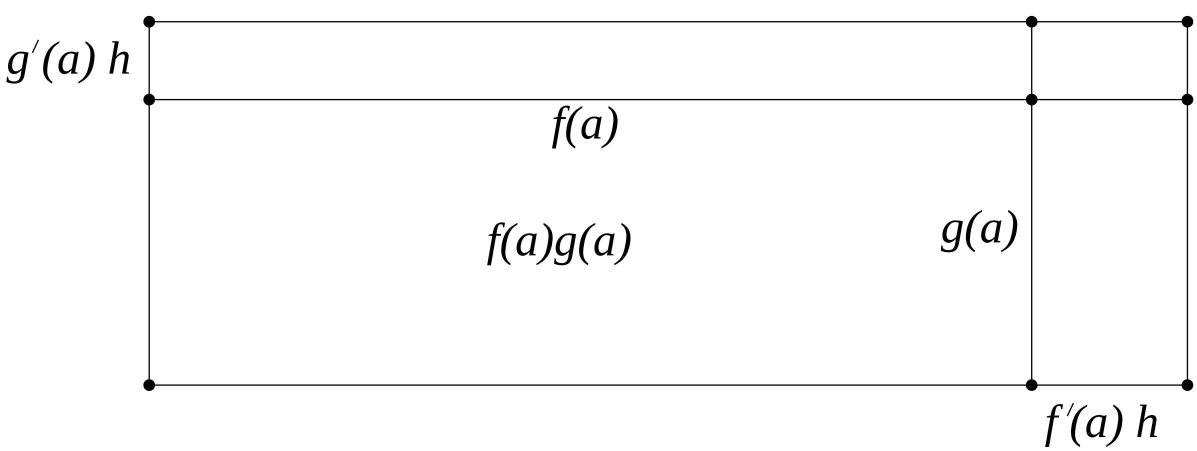
Since $\displaystyle\lim_{h\to 0} \frac{E(h)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}f'(c)h^2}{h} = \frac{1}{2}f'(a) \lim_{h\to 0} h = 0$,

we have $A'(a) = f(a)$.

(Here, I have used the theorem from elementary calculus that all functions have continuous derivatives.)

Why is the product rule true?





$$f(a+h)g(a+h) = f(a)g(a) + f(a)g'(a)h + g(a)f'(a)h + E(h)$$

Why is the chain rule true?