Recursive enumeration of Pythagorean triples

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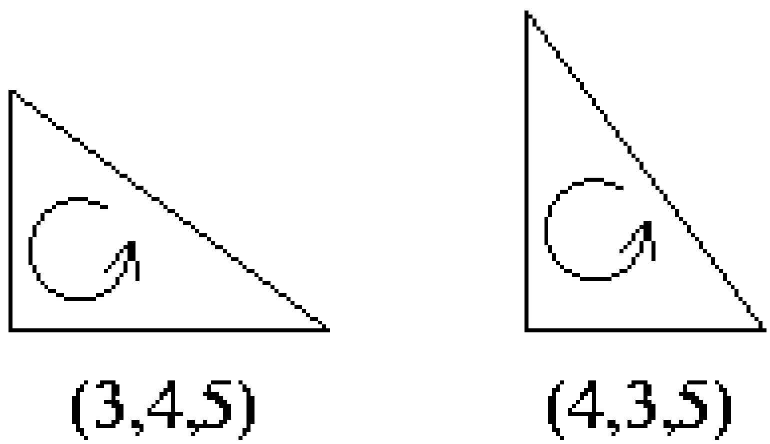
A *Pythagorean triple* (PT) is an ordered triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$.



When a and b are relatively prime, the triple is called a *primitive* PT (PPT).

Each PT is a positive integer multiple of a uniquely determined PPT. For example, if we start with the primitive PT $(8, 15, 17)$ and take its integer multiples, we obtain an infinite sequence of nonprimitive PT's:

$$(16, 30, 34), (24, 45, 51), (32, 60, 68), (40, 75, 85), \ldots$$

Here are some more examples of primitive PT's:

$(11, 60, 61)$	$(12, 35, 37)$	$(171, 140, 221)$	$(252, 115, 277)$
$(15, 112, 113)$	$(16, 63, 65)$	$(207, 224, 305)$	$(288, 175, 337)$
$(23, 264, 265)$	$(24, 143, 145)$	$(279, 440, 521)$	$(360, 319, 481)$
$(27, 364, 365)$	$(28, 195, 197)$	$(315, 572, 653)$	$(396, 403, 565)$
$(35, 612, 613)$	$(36, 323, 325)$	$(387, 884, 965)$	$(468, 595, 757)$
$(39, 760, 761)$	$(40, 399, 401)$	$(423, 1064, 1145)$	$(504, 703, 865)$
$(47, 1104, 1105)$	$(48, 575, 577)$	$(495, 1472, 1553)$	$(576, 943, 1105)$
$(51, 1300, 1301)$	$(52, 675, 677)$	$(531, 1700, 1781)$	$(612, 1075, 1237)$
$(59, 1740, 1741)$	$(60, 899, 901)$	$(603, 2204, 2285)$	$(684, 1363, 1525)$
$(63, 1984, 1985)$	$(64, 1023, 1025)$	$(639, 2480, 2561)$	$(720, 1519, 1681)$
$(71, 2520, 2521)$	$(72, 1295, 1297)$	$(711, 3080, 3161)$	$(792, 1855, 2017)$
$(75, 2812, 2813)$	$(76, 1443, 1445)$	$(747, 3404, 3485)$	$(828, 2035, 2197)$
$(87, 3784, 3785)$	$(88, 1935, 1937)$	$(855, 4472, 4553)$	$(936, 2623, 2785)$
$(95, 4512, 4513)$	$(96, 2303, 2305)$	$(927, 5264, 5345)$	$(1008, 3055, 3217)$
$(99, 4900, 4901)$	$(100, 2499, 2501)$	$(963, 5684, 5765)$	$(1044, 3283, 3445)$

There is a very old method for generating all PPT's (it is sometimes credited to Euclid). You can find a proof in almost any book on elementary number theory.

Take a pair of relatively prime positive integers (m, n) with $m > n$. Put:

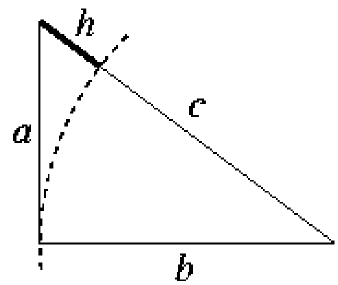
1. $T(m, n) = (m^2 - n^2, 2mn, m^2 + n^2)$ if one of m or n is even.
2. $T(m, n) = \left(\frac{m^2-n^2}{2}, mn, \frac{m^2+n^2}{2}\right)$ if both of m and n are odd.

For example, $T(2, 1) = (3, 4, 5)$ and $T(3, 1) = (4, 3, 5)$.

This enumeration method gives each PPT once, and taking all their multiples gives all the PT's.

Recently, a paper of Peter Wade and William Wade in the *College Math. J.* gave another method for generating PT's.

Define the *height* of (a,b,c) to be $h = c - b$.



Write $h = pq^2$ where q is as large as possible (that is, so that p is not divisible by the square of any prime).

Define $$d = \begin{cases} pq & \text{if } p \text{ is even} \\ 2pq & \text{if } p \text{ is odd.} \end{cases}$$

d is called the *increment.* It can also be described as the smallest positive integer for which d^2 is divisible by $2h$.

Start with $(a_0, b_0) = (h, 0)$. Recursively, define

$$(a_{k+1}, b_{k+1}) = \left(a_k + d\,,\ \frac{d}{h} a_k + b_k + \frac{d^2}{2h}\right) .$$

Then, the $(a_k, b_k, b_k + h)$ with $k \geq 1$ are a list of *all* the PT's of height h.

For example, if $h = 72 = 2^3 \cdot 3^2 = 2 \cdot 6^2$, we have $p = 2$ and $q = 6$, so $d = pq = 12$. This gives

$$\frac{d}{h} = \frac{1}{6} \quad \text{and} \quad \frac{d^2}{2h} = 1 \ ,$$

and the recursion is

$$(a_{k+1}, b_{k+1}) = \Big(a_k + 12\,,\ \tfrac{1}{6}\,a_k + b_k + 1\Big) \ .$$

Start with $(72, 0)$ as (a_0, b_0), and calculate:

$(a_1, b_1) = \Big(72 + 12\,,\ \tfrac{1}{6}\,72 + 0 + 1\Big) = (84, 13)$,

so the first triple of height 72 is $(84, 13, 85)$.

Then take $(84, 13)$ and calculate:

$(a_2, b_2) = \Big(84 + 12\,,\ \tfrac{1}{6}\,84 + 13 + 1\Big) = (96, 28)$,

so the next triple of height 72 is $(96, 28, 100)$.

The next page shows the first 10 triples of height 72.

$$
\begin{array}{ccc}
(72, 0, 72) & & \\
\downarrow & & \\
(84, 13, 85) & & \\
\downarrow & & \\
(96, 28, 100) & = & 4\,(24, 7, 25) \\
\downarrow & & \\
(108, 45, 117) & = & 9\,(12, 5, 13) \\
\downarrow & & \\
(120, 64, 136) & = & 8\,(15, 8, 17) \\
\downarrow & & \\
(132, 85, 157) & & \\
\downarrow & & \\
(144, 108, 180) & = & 36\,(4, 3, 5) \\
\downarrow & & \\
(156, 133, 205) & & \\
\downarrow & & \\
(168, 160, 232) & = & 8\,(21, 20, 29) \\
\downarrow & & \\
(180, 189, 261) & = & 9\,(20, 21, 29) \\
\downarrow & & \\
\cdots & &
\end{array}
$$

The next two pages show the first 40 triples for the heights $h = 1$, 2, 81, and 162, with stars by the primitive ones.

$h=1$	$h=2$
$(3,4,5)^*$	$(4,3,5)^*$
$(5,12,13)^*$	$(6,8,10)$
$(7,24,25)^*$	$(8,15,17)^*$
$(9,40,41)^*$	$(10,24,26)$
$(11,60,61)^*$	$(12,35,37)^*$
$(13,84,85)^*$	$(14,48,50)$
$(15,112,113)^*$	$(16,63,65)^*$
$(17,144,145)^*$	$(18,80,82)$
$(19,180,191)^*$	$(20,99,101)^*$
$(21,220,221)^*$	$(22,120,122)$
$(23,264,265)^*$	$(24,143,145)^*$
$(25,312,313)^*$	$(26,168,170)$
$(27,364,365)^*$	$(28,195,197)^*$
$(29,420,421)^*$	$(30,224,226)$
$(31,480,481)^*$	$(32,255,257)^*$
$(33,544,545)^*$	$(34,288,290)$
$(35,612,613)^*$	$(36,323,325)^*$
$(37,684,685)^*$	$(38,360,362)$
$(39,760,761)^*$	$(40,399,401)^*$
$(41,840,841)^*$	$(42,440,442)$
$(43,924,925)^*$	$(44,483,485)^*$
$(45,1012,1013)^*$	$(46,528,530)$
$(47,1104,1105)^*$	$(48,575,577)^*$
$(49,1200,1201)^*$	$(50,624,626)$
$(51,1300,1301)^*$	$(52,675,677)^*$
$(53,1404,1405)^*$	$(54,728,730)$
$(55,1512,1513)^*$	$(56,783,785)^*$
$(57,1624,1625)^*$	$(58,840,842)$
$(59,1740,1741)^*$	$(60,899,901)^*$
$(61,1860,1861)^*$	$(62,960,962)$
$(63,1984,1985)^*$	$(64,1023,1025)^*$
$(65,2112,2113)^*$	$(66,1088,1090)$
$(67,2244,2245)^*$	$(68,1155,1157)^*$
$(69,2380,2381)^*$	$(70,1224,1226)$
$(71,2520,2521)^*$	$(72,1295,1297)^*$
$(73,2664,2665)^*$	$(74,1368,1370)$
$(75,2812,2813)^*$	$(76,1443,1445)^*$
$(77,2964,2965)^*$	$(78,1520,1522)$
$(79,3120,3121)^*$	$(80,1599,1601)^*$
$(81,3280,3281)^*$	$(82,1680,1682)$

$h=81$	$h=162$
$(99,20,101)^*$	$(180,19,181)^*$
$(117,44,125)^*$	$(198,40,202)$
$(135,72,153)$	$(216,63,225)$
$(153,104,185)^*$	$(234,88,250)$
$(171,140,221)^*$	$(252,115,277)^*$
$(189,180,261)$	$(270,144,306)$
$(207,224,305)^*$	$(288,175,337)^*$
$(225,272,353)^*$	$(306,208,370)$
$(243,324,405)$	$(324,243,405)$
$(261,380,461)^*$	$(342,280,442)$
$(279,440,521)^*$	$(360,319,481)^*$
$(297,504,585)$	$(378,360,522)$
$(315,572,653)^*$	$(396,403,565)^*$
$(333,644,725)^*$	$(414,448,610)$
$(351,720,801)$	$(432,495,657)$
$(369,800,881)^*$	$(450,544,706)$
$(387,884,965)^*$	$(468,595,757)^*$
$(405,972,1053)$	$(486,648,810)$
$(423,1064,1145)^*$	$(504,703,865)^*$
$(441,1160,1241)^*$	$(522,760,922)$
$(459,1260,1341)$	$(540,819,981)$
$(477,1364,1445)^*$	$(558,880,1042)$
$(495,1472,1553)^*$	$(576,943,1105)^*$
$(513,1584,1665)$	$(594,1008,1170)$
$(531,1700,1781)^*$	$(612,1075,1237)^*$
$(549,1820,1901)^*$	$(630,1144,1306)$
$(567,1944,2025)$	$(648,1215,1377)$
$(585,2072,2153)^*$	$(666,1288,1450)$
$(603,2204,2285)^*$	$(684,1363,1525)^*$
$(621,2340,2421)$	$(702,1440,1602)$
$(639,2480,2561)^*$	$(720,1519,1681)^*$
$(657,2624,2705)^*$	$(738,1600,1762)$
$(675,2772,2853)$	$(756,1683,1845)$
$(693,2924,3005)^*$	$(774,1768,1930)$
$(711,3080,3161)^*$	$(792,1855,2017)^*$
$(729,3240,3321)$	$(810,1944,2106)$
$(747,3404,3485)^*$	$(828,2035,2197)^*$
$(765,3572,3653)^*$	$(846,2128,2290)$
$(783,3744,3825)$	$(864,2223,2385)$
$(801,3920,4001)^*$	$(882,2320,2482)$

The proof that Peter and William Wade gave for their recursion formula is a complicated application of the classical enumeration method. Actually, they only gave a complete proof for the cases when h is of the form q^2 or $2q^2$.

In the spring of 2001, I was working with Elizabeth Wade on her senior independent study project. We found a much simpler proof of the recursion formula, and our proof works equally well for all choices of h. We wrote it up as a paper, “Recursive Enumeration of Pythagorean Triples,” which can be downloaded from my website at OU. It will appear in the *College Math. J.*

Some facts about people named Wade:

1. William Wade is a professor of mathematics at the University of Tennessee.

2. Peter Wade is a mathematics teacher at a high school in Tennessee.

3. Elizabeth Wade was a student at OU, with a double major in mathematics and philosophy, and is now a law student at Duke University.

4. William Wade is the father of Peter Wade.

5. Elizabeth Wade is not related to William or Peter Wade.

The McC-Wade proof of the Wade-Wade recursion formula uses a different enumeration of the PT's, which we call the *height-excess enumeration.* After we developed it, we searched for it in the mathematical literature, and were finally able to find it (disguised in much different forms) in two papers published in MAA journals during the 1970's. Also, in the late 1990's it was rediscovered by two other mathematicians, who published it in the *Missouri J. Math. Sci.*

Theorem 1 (The height-excess enumeration) *As one takes all pairs (k, h) of positive integers, the formula*

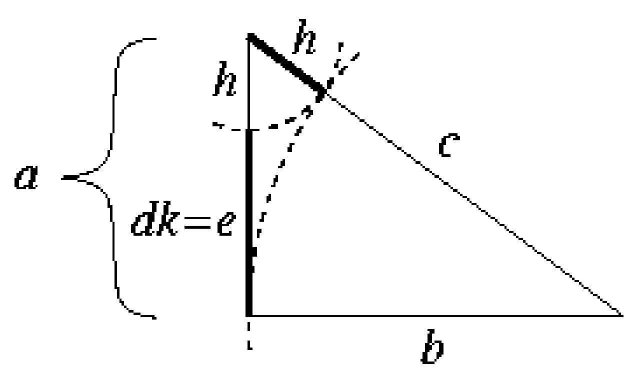
$$P(k,h) = \left(h + dk, dk + \frac{(dk)^2}{2h}, h + dk + \frac{(dk)^2}{2h}\right)$$

produces each Pythagorean triple exactly once.

Notice that h is the height of $P(k, h)$, since

$$c - b = \left(h + dk + \frac{(dk)^2}{2h}\right) - \left(dk + \frac{(dk)^2}{2h}\right) = h$$

Also, notice that $dk = a + b - c$. The number $e = a + b - c$ is called the *excess* of the PT, because it is the extra distance you have to travel, if you go along the two legs of the triangle instead of along the hypotenuse.



Here is how the recursion formula follows from the height-excess enumeration theorem:

The theorem tells us that $P(1,h)$, $P(2,h),\ldots,$ are all the PT's of height h. Write (a_k, b_k, c_k) for $P(k,h)$, so that

$$(a_k, b_k) = \left(h + dk, dk + \frac{(dk)^2}{2h}\right)$$

Using the formula for $P(k+1,h)$, we compute that

$$\begin{aligned}
&(a_{k+1}, b_{k+1}) \\
= \ & \left(h + d(k+1), d(k+1) + \frac{(d(k+1))^2}{2h}\right) \\
= \ & \left(h + dk + d, dk + d + \frac{(dk)^2}{2h} + dk\,\frac{d}{h} + \frac{d^2}{2h}\right) \\
= \ & \left(a_k + d, \frac{d}{h}(h + dk) + \big(dk + \frac{(dk)^2}{2h}\big) + \frac{d^2}{2h}\right) \\
= \ & \left(a_k + d, \frac{d}{h}a_k + b_k + \frac{d^2}{2h}\right)
\end{aligned}$$

That is, the recursion formula just produces $P(k+1,h)$ from $P(k,h)$.

The height-excess enumeration theorem is not difficult to prove. It is just a matter of developing some basic number-theoretic facts about the relation between d and h, and then using college algebra. But the proof is a bit too long to go though here. Feel free to download our paper and read the details.

It turns out that the height-excess enumeration can be used for a lot more than just proving the recursion formula. This seems not to have been realized by its previous discoverers. I have written a paper, "Height and Excess of Pythagorean Triples," which details many uses. Most of these are new and simpler proofs of known theorems about PT's, but some are new results. The paper is available at my website, and will be published in *Mathematics Magazine.*