Complex complex numbers

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To construct the complex numbers, we usually start with the real numbers, and define the complex numbers to be:

$$a + bi, \quad a, b \in \mathbb{R}, \quad i^2 = -1 \ .$$

What happens if we try to construct "complex complex" numbers, by taking:

$$z + wj, \quad z, w \in \mathbb{C}, \quad j^2 = -1 \quad ?$$

This does not work very well, for we then have

$$(i + j)(i - j) = i^2 - j^2 = (-1) - (-1) = 0 \ ,$$

so $1/(i + j)$ could not exist.

But only a small modification is needed to obtain a beautiful number system, the *quaternions:*

$$z + wj, \quad z, w \in \mathbb{C}, \quad j^2 = -1, \quad jz = \overline{z}j \ .$$

This might not be the definition you have seen. Sometimes the quaternions are defined as

$$a + bi + cj + dk, \;\; a, b, c, d \in \mathbb{R},$$
$$i^2 = j^2 = k^2 = -1,$$

plus several multiplication rules for i, j, and k.

That definition violates

Rule # 1 of Quaternions:

Don't ever write k !

(just write ij)

My own uses for quaternions involve only the *unit-length* quaternions:

$$S^3 = \{z + wj \mid z\overline{z} + w\overline{w} = 1\}.$$

The symbol S^3 is used, because these quaternions form the 3-dimensional unit sphere in 4-dimensional space. That is, if we think of $z + wj$ as the complex pair (z, w) in $\mathbb{C}^2 = \mathbb{R}^4$, the condition that $z\overline{z} + w\overline{w} = 1$ says exactly that as a 4-dimensional vector, (z, w) has length 1.

From now on, whenever we say quaternion, we will mean a *unit-length* quaternion.

Under the operation of quaternionic multiplication, S^3 is a (nonabelian) *group*, that is, each element has an inverse in S^3. To see this, define the *bar* of a quaternion $q = z + wj$ to be

$$\overline{q} = \overline{z + wj} = \overline{z} - wj$$

This is in S^3, since $\overline{z}\,\overline{\overline{z}} + (-w)\overline{(-w)} = z\overline{z} + w\overline{w} = 1$. We calculate

$$\begin{gathered} q\,\overline{q} = (z + wj)(\overline{z} - wj) \\ = z\overline{z} - zwj + wzj - w\overline{w}j^2 = z\overline{z} + w\overline{w} = 1 \end{gathered}$$

that is, $\overline{q}$ is q^{-1}. From this, we can deduce the following property of the bar function, without doing any additional computation:

$$\overline{q_1 q_2} = (q_1 q_2)^{-1} = q_2^{-1} q_1^{-1} = \overline{q_2}\ \ \overline{q_1}\ .$$

S^3 is in some ways analogous to the group of S^1 of unit complex numbers,

$$S^1 = \{a + bi \mid a^2 + b^2 = 1\}.$$

Notice that S^1 can be regarded as a subgroup of S^3, it is just the quaternions of the form $z + 0\,j$. From the defining property of j, we have that if $z \in S^1$, then

$$j\,z\,j^{-1} = \overline{z}\,jj^{-1} = \overline{z}$$

As a group, S^3 has a rich geometric and algebraic structure. Today we will examine a bit of this structure.

First, recall that two elements g and h of a group G are *conjugate* if there is a group element k so that $kgk^{-1} = h$.

We have already seen that if $z \in S^1 \subset S^3$, then z and $\overline{z}$ are conjugate:

$$j\,z\,j^{-1} = \overline{z}\,jj^{-1} = \overline{z}$$

We will find a very easy way to determine when any two given elements of S^3 are conjugate.

Define the *real part* of $q = z + wj$ by putting $\Re(q)$ equal to $\Re(z)$, the real part of the complex number z. This makes sense, since then

$$(q+\overline{q})/2 = (z+wj+\overline{z}-wj)/2 = (z+\overline{z})/2 = \Re(z)$$

which agrees with the usual formula $\Re(z) = (z + \overline{z})/2$ for complex numbers.

If two quaternions are conjugate, then they have the same real part. This is because

$$\begin{aligned} \Re(kq\overline{k}) &= (kq\overline{k} + \overline{kq\overline{k}})/2 = (kq\overline{k} + \overline{\overline{k}}\overline{q}\overline{k})/2 \\ &= k\,((q+\overline{q})/2)\,\overline{k} = k\overline{k}\,(q+\overline{q})/2 = \Re(q) \end{aligned}$$

where we used the fact that a real number such as $(q + \overline{q})/2$ commutes with any quaternion.

With just a little more effort, one can show that the converse is true: *if two quaternions have the same real part, then they are conjugate.* Putting these together, we have the

Incredibly Useful Fact: *Two elements of S^3 are conjugate if and only if they have the same real part.*

Since the real part is just the first coordinate in $\mathbb{R}^4$, this says that the conjugacy equivalence classes are exactly the slices of S^3 by the 3-planes $x_1 =$ *constant.*

Notice that this relates **algebraic structure** (conjugacy) to **geometric structure** (the real part).

We will now use the Incredibly Useful Fact to find the conjugacy equivalence classes of elements of S^3 of finite order.

Presumably, everything we will do can be done by direct computation, but that would violate

Rule # 2 of Quaternions:

Use structure to minimize direct computation.

Recall that the *order* of an element g of a group G is the smallest positive n so that $g^n = 1$, or is ∞ if no such n exists.

If two elements of a group are conjugate, they must have the same order, since if $g^n = 1$ then

$$(kgk^{-1})^n = kgk^{-1} \cdot kgk^{-1} \cdot k \ \cdots \ k^{-1} \cdot kgk^{-1}$$
$$= kg^nk^{-1} = k \cdot 1 \cdot k^{-1} = 1 \ .$$

Many elements of S^3 have finite order, for example:

$$-1$$

has order 2,

$$i, \quad j, \quad ij, \quad \text{and} \quad \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$$

have order 4, and

$$\frac{1}{2} + \frac{1}{2}\sqrt{1 - 2\cos\left(\frac{2\pi}{5}\right)}\, i + \cos\left(\frac{\pi}{5}\right) j$$

has order 6.

Elements of order 2*:*

The element -1 has order 2, and it is the *only* element of order 2. For suppose q has order 2. Since $-1 \leq \Re(q) \leq 1$, we have $\Re(q) = \cos(\theta)$ for some θ. By the Incredibly Useful Fact, q is conjugate to the element $\cos(\theta) + \sin(\theta)\, i$ of S^1. The only element of order 2 in S^1 is -1 (because 1 and -1 are the only complex roots of the polynomial $x^2 - 1$), so q is conjugate to -1, that is, $q = k(-1)\overline{k} = -k\overline{k} = -1$.

Elements of order 3*:*

Suppose q has order 3. By the Incredibly Useful Fact, q is conjuate to $\cos(\theta)+\sin(\theta)\, i$, where $\cos(\theta) = \Re(q)$. The only elements of order 3 in S^1 are the cube roots of unity

$$\cos\left(\frac{2\pi}{3}\right) \pm \sin\left(\frac{2\pi}{3}\right) i = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i$$

which are conjugate (same real part). So there is only one conjugacy class of elements of order 3, the elements with real part equal to $-\dfrac{1}{2}$.

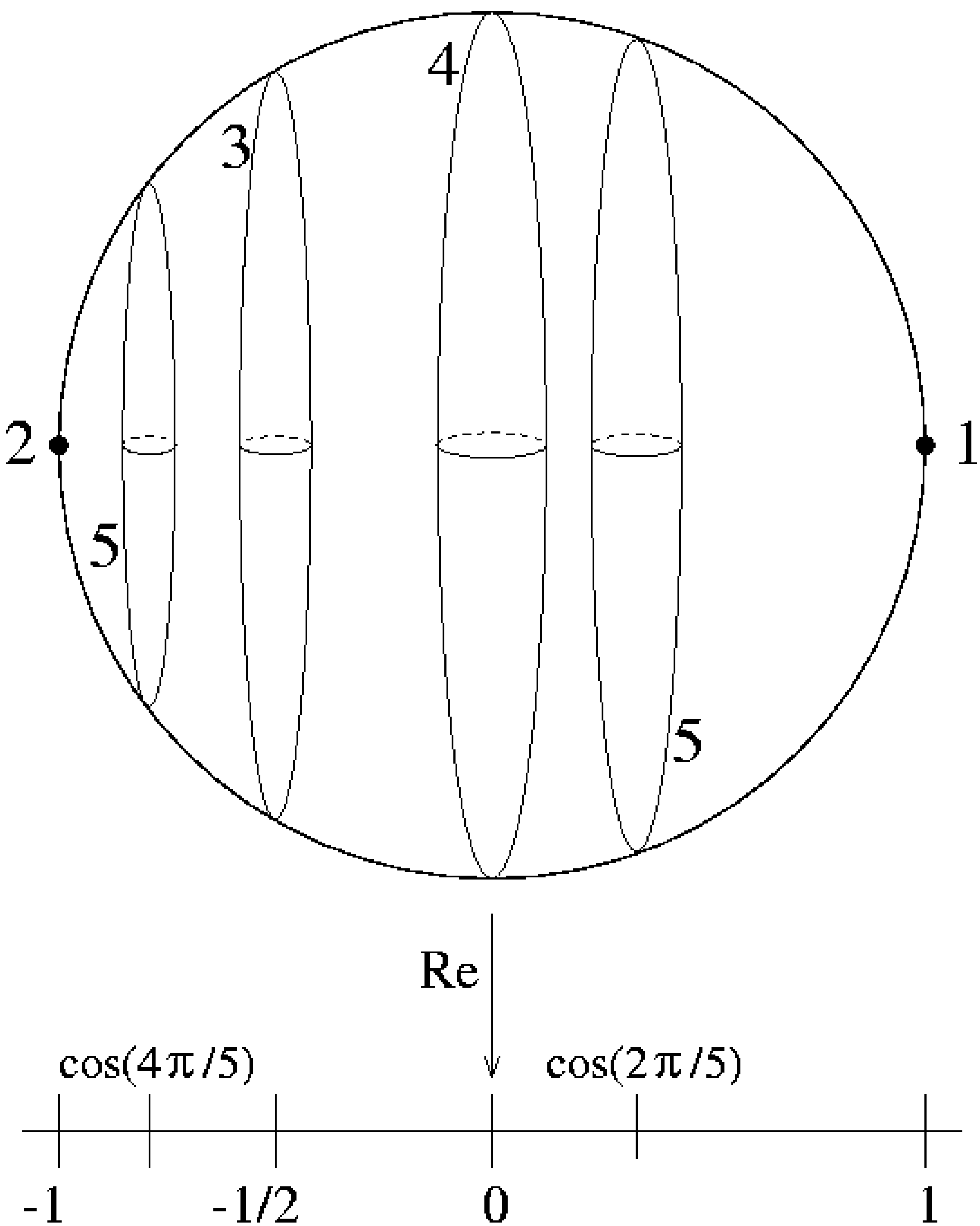
Elements of order n:

Suppose q has order n. By the Incredibly Useful Fact, q is conjuate to $\cos(\theta)+\sin(\theta)\,i$, where $\cos(\theta) = \Re(q)$. The only elements of order n in S^1 are the n^{th} roots of unity that are not roots of unity for a smaller power. So the conjugacy classes of elements of order n in S^3 correspond exactly to the complex conjugate pairs of n^{th} roots of unity that are not roots of unity for a smaller power.

Thus, for example, there is one conjugacy class of elements of order 4, the quaternions that are conjugate to i, and there are two conjugacy classes of elements of order 5, conjugate to

$$\cos\left(\frac{2\pi}{5}\right) \pm \sin\left(\frac{2\pi}{5}\right) i \text{ and } \cos\left(\frac{4\pi}{5}\right) \pm \sin\left(\frac{4\pi}{5}\right) i$$

The next page is a picture of S^3, showing the elements of orders 2, 3, 4, and 5.


4
3
2
1
5
5
Re
cos(4π/5)
cos(2π/5)
-1
-1/2
0
1