Not necessarily algebraic topology

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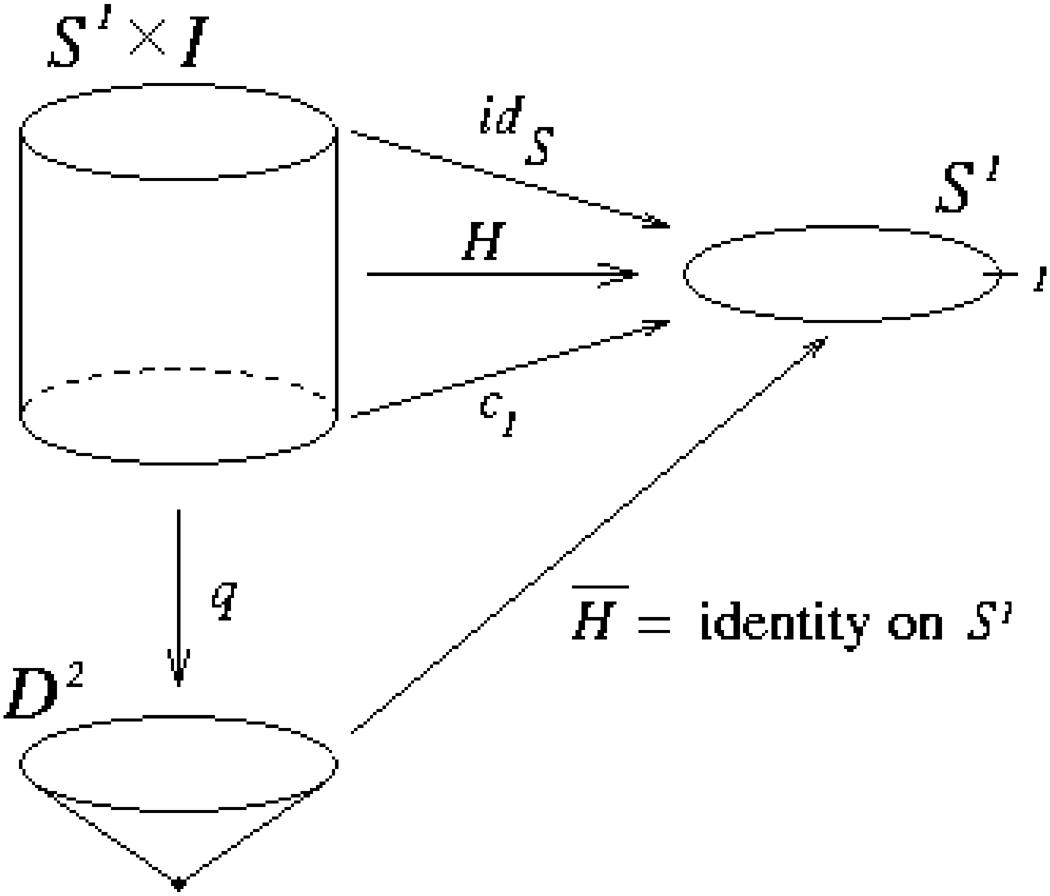
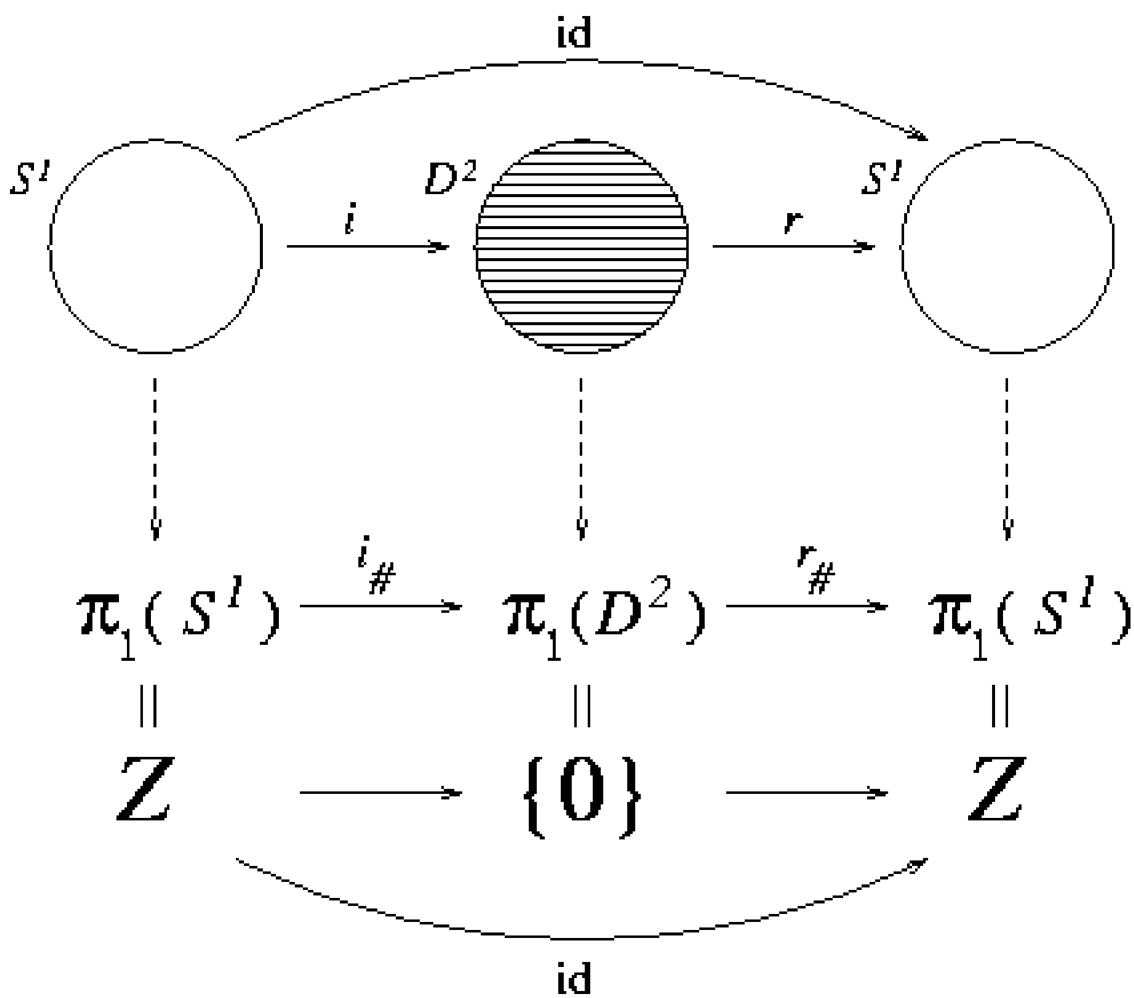
Theorem 1 *S^1 is not contractible (i. e. there is no homotopy from the identity map of S^1 to the constant map).*



Equivalently, there is no map from D^2 to S^1 which is the identity on S^1 (this version is called the *No Retraction Theorem*).

"Easy" algebraic topology proof: If there were a retraction, then by passing to the fundamental groups we would obtain a contradiction.



But there is actually a fairly easy topological proof (that almost certainly goes back to Eilenberg).

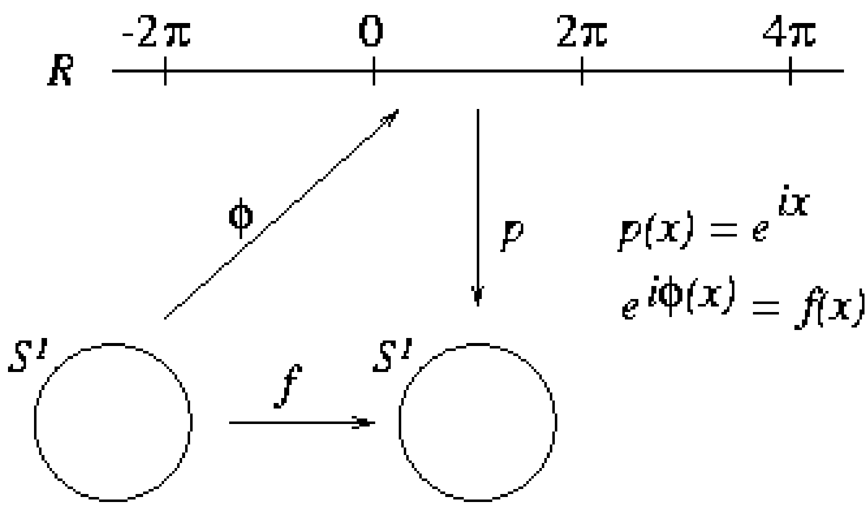
It appears in:

Robert F. Brown, Elementary consequences of the noncontractibility of the circle, *American Mathematical Monthly* 81 (1974), 247-252.

and it is also between the lines in

James Dugundji, *Topology*, Allyn and Bacon, 1966.

Lemma 2 *If $f\colon S^1 \to S^1$ is homotopic to a constant map, then there is a lift $\phi\colon S^1 \to \mathbb{R}$ of f.*



Step 1: Show that if $g\colon S^1 \to S^1$ lifts, and $h\colon S^1 \to S^1$ satisfies $h(x) \neq -g(x)$ for all $x \in S^1$, then h also lifts.

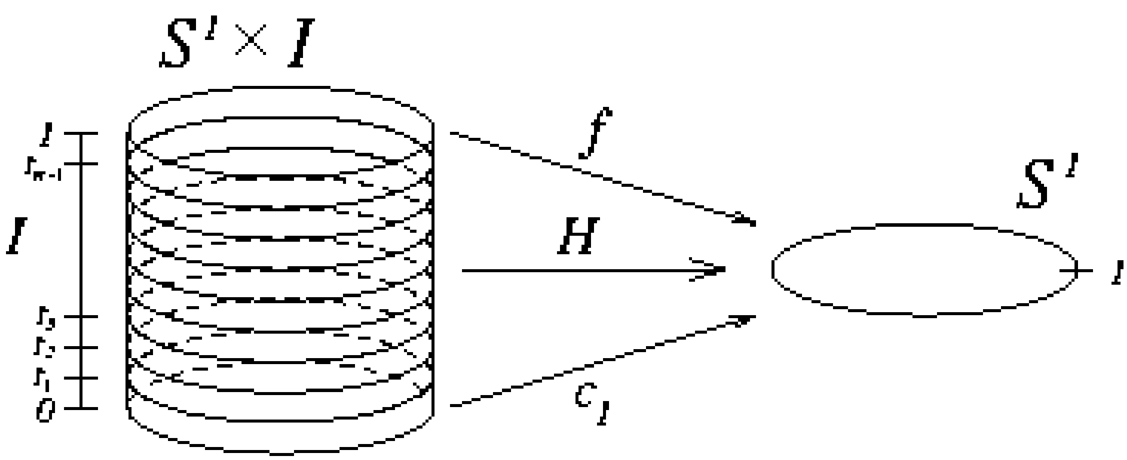
$\frac{h(x)}{g(x)} \in S^1$, and since $\frac{h(x)}{g(x)}$ is never -1, we can write $\frac{h(x)}{g(x)}$ as $e^{i\psi(x)}$ where $\psi(x) \in (-\pi, \pi)$.

If γ is a lift of g, i. e. $g(x) = e^{i\gamma(x)}$, then $\gamma + \psi$ is a lift of h. For we have

$$e^{i(\gamma(x)+\psi(x))} = e^{i\gamma(x)}\, e^{i\psi(x)} = g(x)\,\frac{h(x)}{g(x)} = h(x)\ .$$

Step 2: Complete the proof.

We are assuming that f is homotopic to c_1, and want to show that f lifts to $\mathbb{R}$.



Since H is uniformly continuous, there exists $\delta > 0$ so that if $\| x - y \| < \delta$, then $\| H(x) - H(y) \| < 2$. Choose points t_i in I with $t_{i+1} - t_i < \delta$, then using step 1,

$$
\begin{aligned}
c_1 = H|_{S^1\times\{0\}} &\quad \text{lifts,} \\
H|_{S^1\times\{t_1\}} &\quad \text{lifts,} \\
H|_{S^1\times\{t_2\}} &\quad \text{lifts,} \\
H|_{S^1\times\{t_3\}} &\quad \text{lifts,} \\
\vdots \\
H|_{S^1\times\{t_{n-1}\}} &\quad \text{lifts,} \\
f = H|_{S^1\times\{1\}} &\quad \text{lifts.}
\end{aligned}
$$

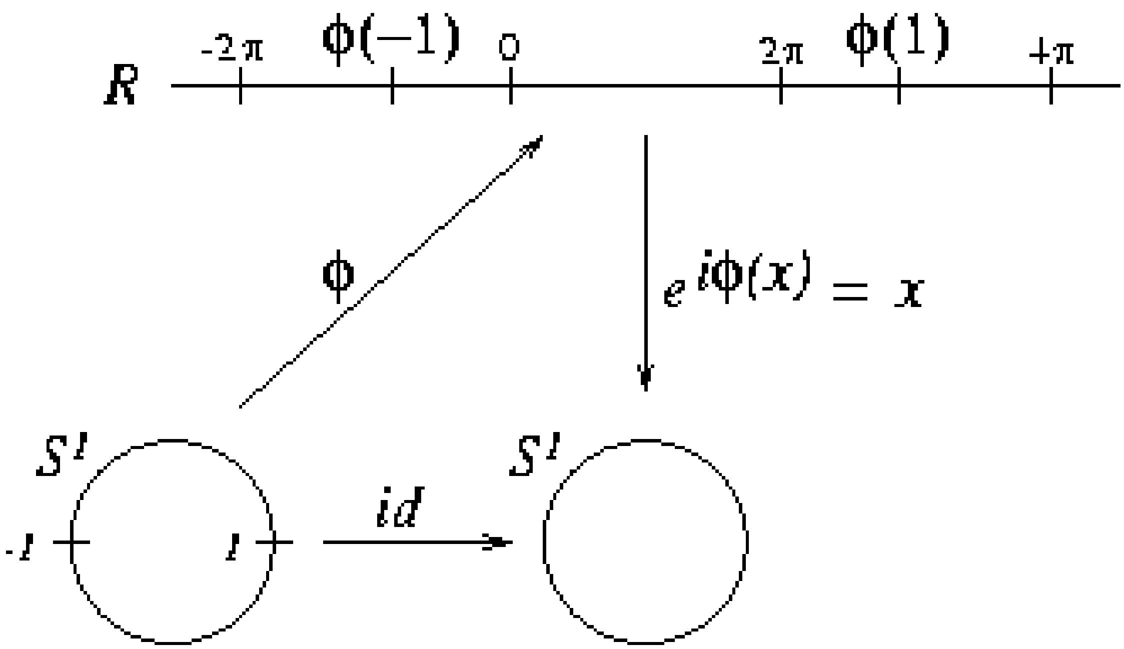
Remarks

1. The converse (that if f lifts, then it is homotopic to a constant) is easy, since $\mathbb{R}$ is contractible.

2. The Lemma is true for $f\colon X^{\text{compact metric}} \longrightarrow S^1$, by the same proof.

Proof that S^1 is not contractible:

Suppose that id_{S^1} were homotopic to a constant map. By the lemma, it would lift to $\mathbb{R}$.



The lift ϕ must be injective, since $e^{i\phi(x)}$ is injective. This violates the Intermediate Value Theorem.

(Choose α between $\phi(-1)$ and $\phi(1)$. Both arcs in S^1 from -1 to 1 must contain a point that maps to α.)

Standard consequences of the noncontractibility of S^1

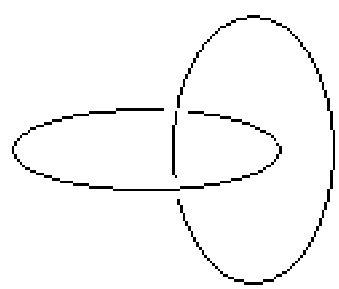
1. The No Retraction Theorem

2. The Brouwer Fixed Point Theorem

3. The Fundamental Theorem of Algebra

Less standard consequences

4. These circles are linked (which implies that the Hopf map from S^3 to S^2 is not homotopic to a constant map, so $\pi_3(S^2) \neq 0$):



5. The complex projectivizing map

$$\begin{array}{c} \mathbb{C}^{n+1} - \{0\} \\ \downarrow \\ \mathbb{CP}^n \end{array} \qquad q(z_0, \ldots, z_n) = [z_0, \ldots, z_n]$$

does not admit a cross-section.

6. The Jordan Curve Theorem

   "An elementary proof can be found in Dugundji, p. 362."