Mathematics 3113-005 Name (please print) ______________________

Exam III Form A

April 29, 2011

Instructions: Read the question carefully and make sure that you answer the question given. Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers.

I. (14) Making use of the tables when needed, calculate the following Laplace transforms and inverse Laplace transforms:

(i) $\mathcal{L}\left(t^{6/5} - e^{-3t}\right)$

$$\frac{\Gamma(11/5)}{s^{11/5}} - \frac{1}{s+3}$$

(ii) $\mathcal{L}(4\cos^2(5t))$ (you may need the identity $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$)

$$\mathcal{L}(4\cos^2(5t)) = \mathcal{L}(2 + 2\cos(10t)) = \frac{2}{s} + \frac{2s}{s^2+100}$$

(iii) $\mathcal{L}(4\cosh^2(2t))$

$$\mathcal{L}(4\cosh^2(2t)) = \mathcal{L}\left(4\left(\frac{e^{2t}+e^{-2t}}{2}\right)^2\right) = \mathcal{L}(e^{4t} + 2 + e^{-4t}) = \frac{1}{s-4} + \frac{2}{s} + \frac{1}{s+4}$$

(iv) $\mathcal{L}\left(x'' + 3x' - 5x\right)$, if $x(0) = 2$, $x'(0) = 3$ (give the answer in terms of $X(s)$, the Laplace transform of $x(t)$).

$$\begin{aligned}\mathcal{L}\left(x'' + 3x' - 5x\right) &= \mathcal{L}(x'') + 3\,\mathcal{L}(x') - 5\,\mathcal{L}(x) = s\,\mathcal{L}(x') - 3 + 3(sX(s) - 2) - 5X(s)\\ &= s(sX(s) - 2) - 3 + 3sX(s) - 6 - 5X(s) = (s^2 + 3s - 5)X(s) - 2s - 9\end{aligned}$$

(v) $f(t)$ if $F(s) = \dfrac{11+s}{s^2+7}$

$$F(s) = \frac{11}{s^2+(\sqrt{7})^2} + \frac{s}{s^2+(\sqrt{7})^2} = \frac{11}{\sqrt{7}}\,\frac{\sqrt{7}}{s^2+(\sqrt{7})^2} + \frac{s}{s^2+(\sqrt{7})^2},\ \text{so } f(t) = \frac{11}{\sqrt{7}}\sin(\sqrt{7}\,t) + \cos(\sqrt{7}\,t).$$

II. (4) For the matrix $P = \begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix}$, one of the eigenvalues is -2, and an eigenvector associated to -2 is $\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$. Use this information to write one specific solution of the first-order homogeneous linear system $X' = PX$.

$$X = e^{-2t}\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$$

III. (12) For a certain homogeneous first-order linear system $X' = PX$ of three equations in three unknown functions, three linearly independent solutions are $X_1 = e^{2t}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $X_2 = e^{4t}\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, and $X_3 = e^{-3t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Write a 3×3 matrix whose determinant is the Wronskian of these three solutions, and calculate the determinant.

Expanding down the middle column, we have

$$W(X_1, X_2, X_3) = \det\begin{bmatrix} e^{2t} & 0 & e^{-3t} \\ -e^{2t} & e^{4t} & 2e^{-3t} \\ -2e^{2t} & -2e^{4t} & e^{-3t} \end{bmatrix}$$

$$= e^{4t}\det\begin{bmatrix} e^{2t} & e^{-3t} \\ -2e^{2t} & e^{-3t} \end{bmatrix} - (-2e^{4t})\det\begin{bmatrix} e^{2t} & e^{-3t} \\ -e^{2t} & 2e^{-3t} \end{bmatrix} = e^{4t}(3e^{-t}) + 2e^{4t}(3e^{-t}) = 9e^{3t}$$

(b) Write a general solution for the system, and use Gauss-Jordan elimination to solve the initial value problem $X' = PX$, $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = 5$ (that is, find the specific solutions x_1, x_2, and x_3 that satisfy the IVP).

A general solution is $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + c_3 e^{-3t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. We want

$$\begin{aligned} x_1(0) &= c_1 + c_3 = 1 \\ x_2(0) &= -c_1 + c_2 + 2c_3 = 0 \\ x_3(0) &= -2c_1 - 2c_2 + c_3 = 5 \end{aligned}$$

Using Gauss-Jordan elimination to solve for c_1, c_2, and c_3, we have

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 1 \\ -1 & 1 & 2 & 0 \\ -2 & -2 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & -2 & 3 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 9 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

so we have $c_1 = 0$, $c_2 = -2$, and $c_3 = 1$. Therefore the solution of the initial value problem is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \cdot e^{2t}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} - 2 \cdot e^{4t}\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-3t} \\ -2e^{4t} + 2e^{-3t} \\ 4e^{4t} + e^{-3t} \end{bmatrix}$$

IV. (6) (a) Give a specific example of three nonzero 2×2 matrices A, B, and C for which $AB = AC$ but $B \neq C$.

There are many possible examples, such as

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(b) Show that if A, B, and C are 2×2 matrices for which $AB = AC$ and $\det(A) \neq 0$, then $B = C$.

When $\det(A) \neq 0$, A has an inverse matrix A^{-1}. So we can multiply by A^{-1} to get $A^{-1}AB = A^{-1}AC$, that is, $IB = IC$, so $B = C$.

V. (3) Define an *eigenvalue* of a matrix A, and define an *eigenvector* associated to that eigenvalue. You may use the version of the definitions given in class, or the version given in the book, or any equivalent statement.

An *eigenvalue* of A is a number λ such that $\det(A - \lambda I) = 0$, or equivalently such that $A\vec{v} = \lambda\vec{v}$ for some nonzero vector $\vec{v}$.

An *eigenvector* associated to the eigenvalue λ is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

VI. (5) Let P be the matrix $\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of P.

$\det(P - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4)$, so the eigenvalues are -1 and 4.

(b) For the *larger* of the two eigenvalues, find an associated eigenvector.

Since $P - 4I = \begin{bmatrix} 2-4 & 3 \\ 2 & 1-4 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$, any $\begin{bmatrix} a \\ b \end{bmatrix}$ satisfying $2a - 3b = 0$ will work. For example, $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is one.

Check (not required): $\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

VII. (7) (a) Write the definition of $\mathcal{L}(f(t))$ as an integral.

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

(b) As you know, for $a \geq 0$ the step function $u_a(t)$ is defined to be 0 for $0 \leq t < a$ and 1 for $a \leq t$. Use the definition to calculate that $\mathcal{L}(u_a(t)) = \dfrac{e^{-as}}{s}$ for $s > 0$. Write the calculation correctly using limits, not treating infinity as a number.

$$\mathcal{L}(u_a(t)) = \int_0^\infty e^{-st} u_a(t)\,dt = \lim_{b\to\infty} \int_0^b e^{-st} u_a(t)\,dt = \lim_{b\to\infty} \int_0^a e^{-st} u_a(t)\,dt + \int_a^b e^{-st} u_a(t)\,dt$$

$$= \lim_{b\to\infty} \int_0^a e^{-st}\cdot 0\,dt + \int_a^b e^{-st}\cdot 1\,dt = \lim_{b\to\infty} 0 + \left(-\frac{1}{s} e^{-st} \Big|_a^b \right) = \lim_{b\to\infty} -\frac{1}{s} e^{-sb} + \frac{1}{s} e^{-sa} = \frac{e^{-sa}}{s}$$

Formulas for the Laplace Transform

$\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$

$\mathcal{L}(t^\alpha) = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, where $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$

$\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$

$\mathcal{L}(\cos(at)) = \dfrac{s}{s^2+a^2}$

$\mathcal{L}(\sin(at)) = \dfrac{a}{s^2+a^2}$

$\mathcal{L}(\cosh(at)) = \dfrac{s}{s^2-a^2}$

$\mathcal{L}(\sinh(at)) = \dfrac{a}{s^2-a^2}$

$\mathcal{L}\left(\dfrac{1}{2a}\,t\sin(at)\right) = \dfrac{s}{(s^2+a^2)^2}$

$\mathcal{L}\left(\dfrac{1}{2a^3}\,(\sin(at) - at\cos(at))\right) = \dfrac{1}{(s^2+a^2)^2}$

$\mathcal{L}(u_a(t)) = \dfrac{e^{-as}}{s}$, where $u_a(r) = 0$ for $r < a$ and $u_a(r) = 1$ for $r > a$

$\mathcal{L}(f^{(n)}(t)) = s^n\,\mathcal{L}(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \dfrac{1}{s}\,\mathcal{L}(f(t))$

$\mathcal{L}(e^{at}f(t)) = F(s-a)$

$\mathcal{L}(t^n f(t)) = (-1)^n\,\dfrac{d^n}{ds^n}\,F(s)$

$\mathcal{L}\left(\dfrac{f(t)}{t}\right) = \int_s^\infty F(\sigma)\,d\sigma$

$\mathcal{L}((f*g)(t)) = F(s)\,G(s)$, where $(f*g)(t) = \int_0^t f(\sigma)g(t-\sigma)\,d\sigma$

$\mathcal{L}(f(t)) = \dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$, if $f(t+p) = f(t)$ for all t

$\mathcal{L}(u_a(t)f(t-a)) = e^{-as}F(s)$

$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s)$, where $u(r) = 0$ for $r < 0$ and $u(r) = 1$ for $r > 0$

$\mathcal{L}(\delta_a(t)) = e^{-as}$, where $\delta_a(t)$ is the Dirac δ function (this is often written using $\delta(t-a)$ to mean $\delta_a(t)$)