

Instructions: Read the question carefully and make sure that you answer the question given. Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers.

**I.** (a) Write a general solution of  $y^{(4)} + 4y'' + 4y = 0$ .

(8) The characteristic polynomial is  $\lambda^4 + 4\lambda^2 + 4 = (\lambda^2 + 2)^2$ , so has roots  $\pm\sqrt{2}i$  and  $\pm\sqrt{2}i$ . A general solution would be

$$c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x) + c_3 x \cos(\sqrt{2}x) + c_4 x \sin(\sqrt{2}x) .$$

(b) The function  $\cos(x) + 3x^2$  satisfies the DE  $y^{(4)} + 4y'' + 4y = \cos(x) + 12x^2 + 24$ . Find a general solution to this DE.

$$c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x) + c_3 x \cos(\sqrt{2}t) + c_4 x \sin(\sqrt{2}x) + \cos(x) + 3x^2$$

**II.** A certain mass-spring system has  $m = 1$ ,  $c = 2$ , and  $k = 3$ , so can be modeled by the equation  $x'' + 2x' + 3x = 0$ , where  $x = x(t)$ .

(a) Find a general solution.

The characteristic polynomial  $\lambda^2 + 2\lambda + 3$  has roots  $-1 \pm \sqrt{2}i$ . A general solution is  $c_1 e^{-t} \cos(\sqrt{2}t) + c_2 e^{-t} \sin(\sqrt{2}t)$ .

(b) What value of  $c$  would make the system critically damped?

The characteristic polynomial  $\lambda^2 + c\lambda + 3$  would have two equal real roots, so we would need  $c^2 - 4 \cdot 1 \cdot 3 = c^2 - 12 = 0$ , that is,  $c = \sqrt{12} = 2\sqrt{3}$ .

(c) Is the system in (a) (with  $c = 2$ ) overdamped, critically damped, or underdamped?

Underdamped.

**III.** Define what it means to say that a collection of functions  $\{y_1, y_2, \dots, y_n\}$  is *linearly independent*.

(3) It means that  $c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$  for constants  $c_i$  only when all the  $c_i$  are 0.  
or  
It means that if  $c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$  for constants  $c_i$ , then all the  $c_i$  are 0.  
or any of various other ways of stating this.

**IV.** Write the function  $-4 \cos(\sqrt{3}t) - 3 \sin(\sqrt{3}t)$  in phase-angle form.

(5) We have  $A = -4$  and  $B = -3$ , so  $B/A = 3/4$  and  $(A, B)$  lies in the third quadrant. Therefore the phase angle is  $\alpha = \pi + \tan^{-1}(3/4)$ . Since  $C = \sqrt{A^2 + B^2} = 5$ , the phase-angle form is  $5 \cos(\sqrt{3}t - \pi - \tan^{-1}(3/4))$ .

V. Recall that in the method of variation of parameters, one uses a trial solution of the form  $y_p = u_1 y_1 + u_2 y_2$ , and if we choose  $u_1$  and  $u_2$  to satisfy the two conditions  $y_1 u_1' + y_2 u_2' = 0$  and  $y_1' u_1 + y_2' u_2 = f(x)$ , then  $y_p$  will be a particular solution. In this problem, we will use the method to find a particular solution to the DE  $y'' - y = -2e^{-x}$ . Some remarks:

- (10) (1) You do not need to give the theoretical derivation of the two equations involving  $u_1'$  and  $u_2'$ . We will simply apply them, with the correct choices of  $y_1$  and  $y_2$ , to find a particular solution for this example.  
 (2) Please be careful to calculate accurately. Miscalculations are likely to make the problem become much more complicated. The correct integrals you need to calculate for this problem are easy ones.  
 (3) Yes, a solution of this DE can be found rather easily using the method of undetermined coefficients, but that is not relevant to this problem, which concerns the method of variation of parameters.
- (a) Write the appropriate form of the trial solution  $y_p = u_1 y_1 + u_2 y_2$  for  $y'' - y = -2e^{-x}$  (that is, what functions would  $y_1$  and  $y_2$  be?)

The functions  $y_1 = e^x$  and  $y_2 = e^{-x}$  are linearly independent solutions of the associated homogeneous equation  $y'' - y = 0$ , so we may take  $y_p = u_1 e^x + u_2 e^{-x}$ .

- (b) Write the appropriate version of the system of two equations  $y_1 u_1' + y_2 u_2' = 0$  and  $y_1' u_1 + y_2' u_2 = f(x)$  for  $y'' - y = -2e^{-x}$ .

The equations become

$$\begin{aligned} e^x u_1' + e^{-x} u_2' &= 0 \\ e^x u_1' - e^{-x} u_2' &= -2e^{-x} \end{aligned}$$

- (c) Solve the system of two equations and use the results to find a particular solution of  $y'' - y = -2e^{-x}$ .

Subtracting the equations gives  $2e^{-x} u_2' = 2e^{-x}$  and hence  $u_2' = 1$ . The first equation becomes

$$e^x u_1' + e^{-x} = 0,$$

giving  $u_1' = -e^{-2x}$ . Integrating gives  $u_1 = e^{2x}/2$  and  $u_2 = x$ , so a particular solution is

$$y = (e^{-2x}/2) \cdot e^x + x \cdot e^{-x} = e^{-x}/2 + x e^{-x}.$$

Checking (not required), we find

$$\begin{aligned} y' &= -e^{-x}/2 + e^{-x} - x e^{-x} = e^{-x}/2 - x e^{-x} \\ y'' &= -e^{-x}/2 - e^{-x} + x e^{-x} = -3e^{-x}/2 + x e^{-x} \\ y'' - y &= -2e^{-x} \end{aligned}$$

**VI.** (a) Define a *linear combination* of functions.

(8) A linear combination of the functions  $y_1, \dots, y_n$  is a function that is a sum  $c_1y_1 + c_2y_2 + \dots + c_ny_n$ , where the  $c_i$  are numbers.

(b) State the Principle of Superposition for a DE of order  $n$ . Be sure to tell the requirements on the DE (that is, the hypotheses) needed for the Principle of Superposition to apply.

The Principle of Superposition states that for a homogeneous linear DE, any linear combination of solutions is a solution.

(c) Verify the Principle of Superposition for the first-order linear homogeneous DE  $y' + P(x)y = 0$ .

We must show that any linear combination of solutions is a solution. Suppose that  $y_1$  and  $y_2$  are any two solutions. They satisfy  $y_1' + P(x)y_1 = 0$  and  $y_2' + P(x)y_2 = 0$ . Testing an arbitrary linear combination  $Ay_1 + By_2$ , we find

$$\begin{aligned} (Ay_1 + By_2)' + P(x)(Ay_1 + By_2) &= Ay_1' + By_2' + P(x)Ay_1 + P(x)By_2 \\ &= A(y_1' + P(x)y_1) + B(y_2' + P(x)y_2) = 0, \end{aligned}$$

so  $Ay_1 + By_2$  is also a solution.

**VII.** Using the trial solution  $y_p = Ax \cos(x) + Bx \sin(x)$ , carry out the rest of the method of undetermined coefficients to find a particular solution of  $y'' + y = \cos(x)$ .

We have  $y_p' = A \cos(x) + B \sin(x) - Ax \sin(x) + Bx \cos(x)$  and  $y_p'' = -2A \sin(x) + 2B \cos(x) - Ax \cos(x) - Bx \sin(x)$ , so  $y_p'' + y_p = -2A \sin(x) + 2B \cos(x)$ . To equal  $\cos(x)$ , we must have  $A = 0$  and  $B = 1/2$ , giving  $y_p = x \sin(x)/2$ .

**VIII.** For the following two DE's, a general solution of the associated homogeneous equation is shown (you do not need to show how to find it or check that it is a solution). Using this information, write a trial solution for the method of undetermined coefficients, but do *not* continue on to try to determine the exact coefficients.

(a)  $y'' + 4y = \cos(2x)$ ,  $y_c = c_1 \cos(2x) + c_2 \sin(2x)$ .

The general form  $P_m(x)e^{rx} \cos(kx)$  becomes just  $1 \cdot e^{0x} \cdot \cos(2x)$ , so the general form of the trial solution is  $x^s(A \cos(2x) + B \sin(2x))$ . The smallest power of  $s$  that will prevent any term from being a complementary function is  $s = 1$ . Therefore a trial solution is  $Ax \cos(2x) + Bx \sin(2x)$ .

(b)  $y^{(4)} + 4y'' = x^2 e^x \cos(2x)$ ,  $y_c = c_1 + c_2x + c_3 \cos(2x) + c_4 \sin(2x)$ .

The general form  $P_m(x)e^{rx} \cos(kx)$  becomes  $P_2(x)e^x \cdot \cos(2x)$ , so the general form of the trial solution is  $x^s e^x ((A_0 + A_1x + A_2x^2) \cos(2x) + (B_0 + B_1x + B_2x^2) \sin(2x))$ . None of the terms is a complementary function, so  $s = 0$ . Therefore a trial solution is  $e^x ((A_0 + A_1x + A_2x^2) \cos(2x) + (B_0 + B_1x + B_2x^2) \sin(2x))$ ,