Mathematics 3113-005 Name (please print) ____________________

Exam I Form A

February 18, 2011

Instructions: Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers, so if you find yourself involved in a lengthy calculation, it might be a good idea to move on and come back to that problem if you have time.

I. (4) For the first-order linear homogeneous DE $y' + P(x)y = 0$, verify that if y_1 and y_2 are solutions, then so is $Ay_1 + By_2$ for any constants A and B.

II. (16) For each of the following first-order DE's, carry out a substitution to put the DE into a form that can be solved by either separation of variables or the method for linear equations, and simplify. If it is separable, write it as an equality of a differential of v and a differential of x (that is, up to the step where you are about to integrate both sides), and if it is linear, find the integrating factor and multiply through to make the left-hand side a derivative. In either case, *do not continue on from there to find the solution.*

(a) $3xy' + 6y = \sqrt{y}/x$

(b) $xy' = y + \sqrt{x^2 + y^2}$

III. (5) (a) Give a definition of an (ordinary) *differential equation.*

(b) Define the *order* of a differential equation.

(c) Give the general form (not a specific example) of a *first-order initial value problem* (of the kind that appears in the first Existence and Uniqueness Theorem).

IV. (10) A small lake contains 100 billion liters of water, contaminated with a pollutant at 1% concentration. Starting at time $t = 0$, three billion liters per day of water with the pollutant at 4% concentration enters, mixes instantaneously with all the water in the lake, and the same amount of mixed water leaves. We want to know how many days it will take until the level of contamination reachs 2%. Let $x(t)$ denote the volume of pollutant in the lake, in billions of liters.

(a) What is $x(0)$? In terms of $x(t)$, what does the problem ask us to find?

(b) Write an expression for the volume of pollutant that enters between t and $t + \Delta t$.

(c) Write an expression for the approximate volume of pollutant that leaves between t and $t + \Delta t$.

(d) Use (b), (c), and first-semester calculus to set up a first-order linear initial value problem that models the problem, but *do not* go on to solve it or attempt to find the answer to the problem.

V. (10) The differential equation in the following initial value problem is separable. Use the method of separation of variables to find the solutions, and check your solution.

$$y' = 8e^{2x-y},\ y(1) = 2\ .$$

Hint: To get started, recall that $e^{2x-y} = e^{2x}e^{-y}$.

VI. (5) Show that the substitution $v = \ln(y)$ transforms the differential equation $dy/dx + P(x)y = Q(x)y\ln(y)$ into the linear equation $dv/dx + P(x) = Q(x)v(x)$.

VII. (4) (a) From the viewpoint of existence and uniqueness, what is noteworthy about the nonzero solutions of the DE $y' = y^2$?

(b) From the viewpoint of existence and uniqueness, what is noteworthy about the solutions of the Torricelli equation $\dfrac{dV}{dt} = -k\sqrt{y}$?