Mathematics 3113-005 Name (please print) ______________________________

Quiz 7 Form A

April 15, 2011

Instructions: Give concise answers, but clearly indicate your reasoning.

I. The system of linear equations
(3)

$$\begin{aligned} c_1 + c_2 \qquad &= 10 \\ c_1 - c_2 - c_3 &= -1 \\ c_1 \qquad + c_3 &= 12 \end{aligned}$$

arises in solving one of the homework problems (5.1#26). Use the method of Gauss-Jordan elimination to solve this system. That is, rewrite the system as an "augmented" matrix, then do elementary row operations to obtain the values of c_1, c_2, and c_3 that satisfy the system. The first step, writing the augmented matrix, has already been carried out below, just continue the process from there.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 1 & -1 & -1 & -1 \\ 1 & 0 & 1 & 12 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 1 & -1 & -1 & -1 \\ 1 & 0 & 1 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & -2 & -1 & -11 \\ 0 & -1 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 0 & -3 & -15 \\ 0 & -1 & 1 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & -1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right] ,$$

so the solution is $(c_1, c_2, c_3) = (7, 3, 5)$.

II. For the system $X' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} X$, a general solution is
(3)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(do not derive this or check this). Write a system of linear equations whose solution (c_1, c_2, c_3) gives x_1, x_2, and x_3 satisfying $x_1(1) = 0$, $x_2(1) = -1$, $x_3(1) = 7$. *Do not solve this system,* just write it down.

Saying that this is a general solution means we can write any solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_3 e^{2t} \\ c_2 e^{-t} + c_3 e^{2t} \\ -c_1 e^{-t} - c_2 e^{-t} + c_3 e^{2t} \end{bmatrix}$$

for some choice of $c_!$, c_2, and c_3. Specializing to $t = 1$, we want

$$\begin{bmatrix} c_1 e^{-1} + c_3 e^{2} \\ c_2 e^{-1} + c_3 e^{2} \\ -c_1 e^{-1} - c_2 e^{-1} + c_3 e^{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 7 \end{bmatrix} ,$$

giving the conditions

$$c_1 e^{-1} + c_3 e^2 = 0$$

$$c_2 e^{-1} + c_3 e^2 = -1$$

$$-c_1 e^{-1} - c_2 e^{-1} + c_3 e^2 = 7$$

III. (3) Define an *eigenvalue* of a matrix A, and define an *eigenvector* associated to that eigenvalue. You may use the version of the definitions given in class, or the version given in the book, or any equivalent statement.

An *eigenvalue* of A is a number λ such that $\det(A - \lambda I) = 0$, or equivalently such that $A\vec{v} = \lambda\vec{v}$ for some nonzero vector $\vec{v}$.

An *eigenvector* associated to the eigenvalue λ is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

IV. (7) (a) Show how to calculate that the eigenvalues of the matrix $P = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$ are -1 and 6.

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 6$. Its roots -1 and 6 are the eigenvalues.

(b) An eigenvector associated to the eigenvalue -1 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (do not calculate this or check it). Use this to write out a solution X_1 of the system $X' = PX$.

A corresponding solution is $X_1 = e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$.

(c) For the eigenvalue 6, find an associated eigenvector $\begin{bmatrix} a \\ b \end{bmatrix}$.

We must solve the system with augmented matrix

$$\left[\begin{array}{cc|c} 3-6 & 4 & 0 \\ 3 & 2-6 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -3 & 4 & 0 \\ 3 & -4 & 0 \end{array}\right] .$$

Using Gauss-Jordan elimination, we have

$$\left[\begin{array}{cc|c} -3 & 4 & 0 \\ 3 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -3 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

which says that $3a - 4b = 0$ or $a = 4b/3$. We may take $b = 3$, giving $a = 4$, so one eigenvector is $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. [Any nonzero scalar multiple of this is also an eigenvector associated to the eigenvalue 6.]

Check (not required):

$$\begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 12+12 \\ 12+6 \end{bmatrix} = \begin{bmatrix} 24 \\ 18 \end{bmatrix} = 6 \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} ,$$

so $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is indeed an eigenvector associated to the eigenvalue 6.