Mathematics 3113-005 Name (please print) ____________________

Final Exam Form B

May 12, 2011

Instructions: Read the question carefully and make sure that you answer the question given. Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers.

I. (12) Making use of the tables when needed, calculate the following Laplace transforms and inverse Laplace transform:

(i) $\mathcal{L}\left(\dfrac{e^t - e^{-t}}{t}\right)$ (remember that $\ln(a) - \ln(b) = \ln(\frac{a}{b})$, and $\lim_{b\to\infty} \dfrac{b-1}{b+1} = 1$)

(ii) $\mathcal{L}(t\sin(t))$

(iii) $f(t)$, if $F(s) = \dfrac{2s}{s^2 + 4s + 13}$

II. (5) For the rational function $\dfrac{\lambda^3 + 1}{(\lambda^2 - 9)^2(\lambda^2 + 9)^2}$, write the sum of partial fractions with unknown coefficients that would be used in the method of partial fractions, but *do not* go on to solve for the coefficients.

III. (8) Use an integrating factor to solve the linear IVP

$$y' = (1 - y)\cos(x), \quad y(\pi) = 3 \ .$$

You will want to start by putting the DE into the standard form for a first-order linear DE.

IV. (4) Define an *eigenvalue* of a matrix A, and define an *eigenvector* associated to that eigenvalue. You may use the version of the definitions given in class, or the version given in the book, or any equivalent statement.

V. (5) (a) Define a *linear combination* of functions.

(b) State the Principle of Superposition for a DE of order n. Be sure to tell the requirements on the DE (that is, the hypotheses) needed for the Principle of Superposition to apply.

VI. (8) For the matrix $\begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix}$, the eigenvalues are 3, 1, and -2. An eigenvector associated to 3 is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, an eigenvector associated to 1 is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and an eigenvector associated to -2 is $\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$.

(a) Write a general solution to the system $X' = \begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix} X$.

(b) Write a set of linear equations whose solutions are the unknown coefficients in the general solution if the initial values are $x_1(-1) = -3$, $x_2(-1) = 0$, and $x_3(-1) = 3$.

(c) Write an augmented matrix which would be the first step in using Gauss-Jordan elimination to solve the system in part (b), but *do not* continue with the process or attempt to find the unknown coefficients or the solution to the differential equation.

VII. (a) Give a specific example of three nonzero 2×2 matrices A, B, and C for which $AB = AC$ but $B \neq C$.
(6)
(b) Show that if A, B, and C are 2×2 matrices for which $AB = AC$ and $\det(A) \neq 0$, then $B = C$.

VIII. In this problem, we will solve the initial value problem $x'' - 4x' - 5x = 0$, $x(0) = x'(0) = 1$.
(8)
(a) Use the characteristic polynomial to write down a general solution with coefficients c_1 and c_2.

(b) Use the initial conditions to write down a system of two linear equations that c_1 and c_2 must satisfy.

(c) Use Gauss-Jordan elimination to solve the system of two linear equations, and *write the solution of the initial value problem.*

IX. In this problem, we will solve the initial value problem $x'' - 4x' - 5x = 0$, $x(0) = x'(0) = 1$.
(10)
(a) Apply the Laplace transform to change the problem to an algebra equation for $X(s)$, the Laplace transform of $x(t)$. Solve it for $X(s)$ to obtain an expression giving $X(s)$ as a rational function of s.

(b) Write $X(s)$ as a sum of partial fractions, with unknown coefficients, and find the coefficients.

(c) Apply the inverse transform to find the solution $x(t)$.

X. In this problem, we will solve the differential equation $x'' - 4x' - 5x = 0$.
(14)
(a) Rewrite the DE $x'' - 4x' - 5x = 0$ as an equivalent first-order system with unknown functions x and $y = x'$ (or you may write $x_1 = x$ and $x_2 = x'$ if you prefer to use that notation).

(b) Write the system in the form $X' = PX$, where P is a 2×2 matrix and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

(c) Find the eigenvalues of P.

(d) An eigenvector associated to one of the eigenvalues is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find an eigenvector associated to the other eigenvalue.

(e) Use the eigenvalues and eigenvectors to write two solutions of $X' = PX$, and use them to write a general solution for X (its top function x will be a general solution $x(t)$ for the DE $x'' - 4x' - 5x = 0$, although not necessarily written in exactly the same way as the general solution found by other methods, and its bottom function y should be $x'(t)$).