Mathematics 4853-001 Name (please print) ____________________

Examination III

April 30, 2010

Instructions: Give concise answers, but clearly indicate your reasoning in any arguments that you give. You do not need to verify commonly used facts, unless demonstrating that you have the ability to verify a fact is the point of the problem.

I. (12) Let X be $\mathbb{R}$ with the lower limit topology.

(a) Prove that X is not connected.

Let $U = (-\infty, 1/2)$ and $V = [1/2, \infty)$. These are disjoint, open in X, nonempty, and their union is X, so X is not connected.

(b) Prove that in X, the sequence $\{1/n\}$ converges to 0.

Let U be any neighborhood of 0. Then U contains some interval $[0, \epsilon)$. If $n > 1/\epsilon$, then $1/n \in [0, \epsilon) \subseteq U$.

(c) Prove that in X, the sequence $\{1 - 1/n\}$ does not converge to 1.

The neighborhood $[1, 2)$ contains none of the points $1 - 1/n$, so the sequence does not converge to 1.

II. (6) Let $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection of closed subsets of a topological space X.

(a) Using the definition of closed set, prove that their intersection $\cap_{\alpha\in\mathcal{A}} C_\alpha$ is closed.

Each $X - C_\alpha$ is open, so $X - \cap_{\alpha\in\mathcal{A}} C_\alpha = \cup_{\alpha\in\mathcal{A}}(X - C_\alpha)$ is open. Therefore $\cap_{\alpha\in\mathcal{A}}$ is closed.

(b) Give an example for which the union $\cup_{\alpha\in\mathcal{A}} C_\alpha$ is not closed.

Let $C_n = [1/(n+1), 1/n]$ for $n \in \mathbb{N}$. These are closed in $\mathbb{R}$. But $\cup_{n=1}^{\infty} C_n = (0, 1]$, which is not a closed subset of $\mathbb{R}$.

III. (6) Take as known the fact that $x \in \overline{A}$ if and only if every neighborhood of x contains a point of A. Prove that if $f\colon X \to Y$ is continuous and $S \subseteq X$, then $f(\overline{S}) \subseteq \overline{f(S)}$.

Let $y \in f(\overline{S})$, say $y = f(x)$ with $x \in \overline{S}$. Let U be any neighborhood of y. Then $f^{-1}(U)$ is open and contains x. Since $x \in \overline{S}$, $f^{-1}(U)$ contains a point s of S. So $f(s) \in U$. Therefore every neighborhood of y contains a point of $f(S)$, so $y \in \overline{f(S)}$.

A different and nice approach that some people used: Since $\overline{f(S)}$ is closed and f is continuous, $f^{-1}(\overline{f(S)})$ is closed. Since $S \subseteq f^{-1}(\overline{f(S)})$, $\overline{S} \subseteq f^{-1}(\overline{f(S)})$. Therefore $f(\overline{S}) \subseteq \overline{f(S)}$.

IV. (6) Let X be compact, and let $f\colon X \to \mathbb{R}$ be continuous. Use the definition of compactness to prove that f is bounded.

Since $X = f^{-1}(\mathbb{R}) = f^{-1}(\cup_{n\in\mathbb{N}}(-n, n)) = \cup_{n\in\mathbb{N}} f^{-1}((-n, n))$, so the collection $\{f^{-1}((-n, n))\}_{n\in\mathbb{N}}$ is an open cover of X. Let $\{f^{-1}((-n_1, n_1)), \ldots, f^{-1}((-n_k, n_k))\}$ be a finite subcover, so $X = \cup_{i=1}^{k} f^{-1}((-n_i, n_i))$. That is, $f(X) \subseteq \cup_{i=1}^{k}(-n_i, n_i) = (-N, N)$ where $N = \max_{i=1}^{k}\{n_i\}$. This says that f is bounded.

V. (6) Prove that if X and Y are Hausdorff spaces, then $X \times Y$ is Hausdorff. (Hint: if $(x, y) \neq (x', y')$, then at least one of $x \neq x'$ or $y \neq y'$ is true. So either x and x' have disjoint neighborhoods in X, or y and y' have disjoint neighborhoods in Y.)

Since $(x, y) \neq (x', y')$, either x and x' have disjoint neighborhoods in X, or y and y' have disjoint neighborhoods in Y. We may assume that x and x' have disjoint neighborhoods U and V, the other case being similar. We then have $(x, y) \in \pi_X^{-1}(U)$ and $(x', y) \in \pi_X^{-1}(V)$, which are open since π_X is continuous, and are disjoint since U and V are disjoint.

VI. (6) Let $\{1, 2, 3\}$ have the discrete topology. Suppose that X is a space and there exists a continuous surjection from X to $\{1, 2, 3\}$. Prove that X is not connected.

One proof: Let $f\colon X \to \{1, 2, 3\}$ be a continuous surjection. Put $U = f^{-1}(\{1\})$ and $V = f^{-1}(\{2, 3\})$. These are 1) open, since f is continuous, 2) disjoint, since $\{1\}$ and $\{2, 3\}$ are disjoint, 3) nonempty, since f is surjective, and their union is X. So X is not connected.

A nicer proof: Let $f\colon X \to \{1, 2, 3\}$ be a continuous surjection. Define $g\colon \{1, 2, 3\} \to \{1, 2\}$ by $g(1) = 1$, $g(2) = 2$, and $g(3) = 2$. Then g is a continous surjection, so the composition $g \circ f\colon X \to \{1, 2\}$ is a continuous surjection. Therefore X is not connected.

VII. (6) Let $f\colon X_1 \to X_2$ and $g\colon Y_1 \to Y_2$ be continuous maps. Define $F\colon X_1 \times Y_1 \to X_2 \times Y_2$ by $F((x, y)) = (f(x), g(y))$. Prove that F is continuous.

$\pi_{X_2} \circ F\colon X_1 \times Y_1 \to X_2$ sends (x, y) to $f(x)$, so it equals $f \circ \pi_{X_1}$. Similarly, $\pi_{Y_2} \circ F\colon X_1 \times Y_1 \to Y_2$ equals $g \circ \pi_{Y_1}$. Since these are continuous, the Mapping Into Products Theorem shows that F is continuous.

or: Let $U \times V$ be a basic open set in $X_2 \times Y_2$. Then $F^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$, since $(f(x), g(y))$ is in $U \times V$ exactly when $f(x)$ is in U and $g(x)$ is in V. Since f and g are continuous, $f^{-1}(U)$ and $g^{-1}(V)$ are open, so $F^{-1}(U \times V)$ is open.

VIII. (6) Let (X, d) be a metric space, and let $x \in X$. Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ has the property that for every n, $x_n \in B(x, 1/n)$. Prove that $\{x_n\} \to x$.

Let U be a neighborhood of x. Then there exists $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$. For all $n > 1/\epsilon$, $d(x, x_n) < 1/n < \epsilon$, so $x_n \in B_d(x, \epsilon) \subseteq U$.