Mathematics 4853-001 Name (please print) ____________________

Examination III

April 30, 2010

Instructions: Give concise answers, but clearly indicate your reasoning in any arguments that you give. You do not need to verify commonly used facts, unless demonstrating that you have the ability to verify a fact is the point of the problem.

I. (12) Let X be $\mathbb{R}$ with the lower limit topology.

(a) Prove that X is not connected.

(b) Prove that in X, the sequence $\{1/n\}$ converges to 0.

(c) Prove that in X, the sequence $\{1-1/n\}$ does not converge to 1.

II. (6) Let $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection of closed subsets of a topological space X.

(a) Using the definition of closed set, prove that their intersection $\cap_{\alpha\in\mathcal{A}}C_\alpha$ is closed.

(b) Give an example for which the union $\cup_{\alpha\in\mathcal{A}}C_\alpha$ is not closed.

III. (6) Take as known the fact that $x \in \overline{A}$ if and only if every neighborhood of x contains a point of A. Prove that if $f\colon X \to Y$ is continuous and $S \subseteq X$, then $f(\overline{S}) \subseteq \overline{f(S)}$.

IV. (6) Let X be compact, and let $f\colon X \to \mathbb{R}$ be continuous. Use the definition of compactness to prove that f is bounded.

V. (6) Prove that if X and Y are Hausdorff spaces, then $X \times Y$ is Hausdorff. (Hint: if $(x, y) \neq (x', y')$, then at least one of $x \neq x'$ or $y \neq y'$ is true. So either x and x' have disjoint neighborhoods in X, or y and y' have disjoint neighborhoods in Y.)

VI. (6) Let $\{1, 2, 3\}$ have the discrete topology. Suppose that X is a space and there exists a continuous surjection from X to $\{1, 2, 3\}$. Prove that X is not connected.

VII. (6) Let $f\colon X_1 \to X_2$ and $g\colon Y_1 \to Y_2$ be continuous maps. Define $F\colon X_1 \times Y_1 \to X_2 \times Y_2$ by $F((x, y)) = (f(x), g(y))$. Prove that F is continuous.

VIII. (6) Let (X, d) be a metric space, and let $x \in X$. Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ has the property that for every n, $x_n \in B(x, 1/n)$. Prove that $\{x_n\} \to x$.