Mathematics 4853-001 Name (please print) ____________________

Final Examination

May 12, 2010

Instructions: Give concise answers, but clearly indicate your reasoning in any arguments that you give. You do not need to verify commonly used facts, unless demonstrating that you have the ability to verify a fact is the point of the problem.

I. (4) Let X be a set. Give a full, precise definition of a *topology on X*.

II. (6) Let X be a topological space.

(a) Let U be a subset of X. Suppose that for every $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$. Prove that U is open.

(b) Let Y be a topological space, and let $f\colon X \to Y$ be a function. Suppose that for every $x \in X$ and every neighborhood U of $f(x)$, there exists a neighborhood V of x in X such that $f(V) \subseteq U$. Prove that f is continuous.

III. (5) Let S be any set with at least two elements. Using a Cantor-style argument, prove that the product

$$P = S \times S \times S \times \cdots = \{(s_1, s_2, s_3, \ldots) \mid s_i \in S\}$$

of countably many copies of S is uncountable. [Hint: Choose two different elements s and t of S. If $\phi\colon \mathbb{N} \to P$ is any function, use s and t to construct an element $p \in P$ that is not equal to any $\phi(n)$.]

IV. (5) Take as known the fact that if (X, d) is a metric space, then d is continuous. Prove that if $f\colon Z \to X$ and $g\colon Z \to X$ are continuous, then the function $h\colon Z \times Z \to \mathbb{R}$ defined by $h((z_1, z_2)) = d(f(z_1), g(z_2))$ is continuous. [Hint: What is the domain of d?]

V. (6) (a) State the Basis Recognition Theorem.

(b) Let X be a topological space, and let $\mathcal{B}$ be a basis that generates the topology of X. Let $\mathcal{C}$ be a collection of open subsets of X. Suppose that for every *basic* open set $B \in \mathcal{B}$ and every $x \in B$, there exists an element $C \in \mathcal{C}$ such that $x \in C \subseteq B$. Prove that $\mathcal{C}$ is a basis that generates the topology of X.

VI. (5) Let X and Y be two topological spaces, each with the cofinite topology. Let $\phi\colon X \to Y$ be a bijection. Prove that ϕ is a homeomorphism.

VII. (6) Let X be the union of the usual line $\mathbb{R}$ with one additional point p which is not in $\mathbb{R}$. Define a topology on X by the rule that U is open if and only if $p \in U$ and $U \cap \mathbb{R}$ is open (possibly empty) in the standard topology on $\mathbb{R}$. (You do not need to check that this defines a topology.)

(a) Prove that no sequence of points in $\mathbb{R}$ converges to p.

(b) Prove that the constant sequence $\{p_n\}$ for which every $p_n = p$ converges to each point of $\mathbb{R}$.

VIII. (5) Let $f\colon X \to Y$ be a continuous, surjective function. Prove that if X is compact, then Y is compact.

IX. (5) Let X be a compact topological space. Let $f\colon X \to \mathbb{R}$ be a continuous function. Suppose that $f(x) > 0$ for every $x \in X$. Prove that there exists a *positive* number δ such that for every $x \in X$, $\delta \leq f(x)$. [Hint: One approach uses the collection $\{(1/n, n)\}$ of open sets in $\mathbb{R}$. Another approach uses the fact every continuous real-valued function on a compact space is bounded.]

X. (6) Let (X, d) be a metric space, and let $x_0 \in X$. For $\epsilon > 0$, define $C(x, \epsilon) = \{z \in X \mid d(x_0, z) \leq \epsilon\}$. It is called the *closed ball* of radius ϵ centered at x_0.

(a) Prove that $C(x_0, \epsilon)$ is a closed subset of X.

(b) Give an example showing that sometimes the closure $\overline{B(x_0, \epsilon)}$ of $B(x_0, \epsilon)$ in X is not equal to $C(x_0, \epsilon)$.

XI. (6) Let $X = C([0, 1])$, the set of continuous functions from $[0, 1]$ to $\mathbb{R}$. For $f, g \in X$, define $d(f, g) = \int_0^1 |f(x) - g(x)|\, dx$ and $\rho(f, g) = \max_{x\in[0,1]}\{|f(x) - g(x)|\}$. These are metrics (do not verify this). Let f_n be the function $f_n(x) = x^n$, and let 0 be the zero function. Using the definition of convergence, prove that:

(a) In the d-topology, $\{f_n\}$ converges to 0.

(b) In the ρ-topology, $\{f_n\}$ does not converge to 0.

XII. (6) (a) Define for a metric space (X, d), define what it means to say that a sequence $\{x_n\}$ in X is *Cauchy,* and define what it means to say that (X, d) is *complete.*

(b) Prove that if $\{x_n\}$ converges, say $\{x_n\} \to x_0$, then $\{x_n\}$ is Cauchy.

XIII. (10) Let $\{f_n\}$ be a sequence of functions from $[0, 1]$ to $\mathbb{R}$.

(a) Define what it means to say that $\{f_n\}$ converges to $f \colon [0, 1] \to \mathbb{R}$ *uniformly.*

(b) Prove that if $\{f_n\} \to f$ uniformly, then $\{f_n\} \to f$ pointwise.

(c) State without proof an example of continuous functions $f_n \colon [0, 1] \to \mathbb{R}$ and another function f such that $\{f_n\}$ converges to f pointwise, but not uniformly.

(d) State the Uniform Convergence Theorem.