Math 4853 homework

51. (not to turn in) Let X be a set with the cofinite topology. Prove that every subspace of X has the cofinite topology (i. e. the subspace topology on each subset A equals the cofinite topology on A). Notice that this says that every subset is compact. So not every compact subset of X is closed (unless X is finite, in which case X and all of its subsets are finite sets with the discrete topology).

52. (4/28) Let X be a Hausdorff space. Prove that every compact subset A of X is closed. Hint: Let $x \notin A$. For each $a \in A$, choose disjoint open subsets U_a and V_a of X such that $x \in U_a$ and $a \in V_a$. The collection $\{V_a \cap A\}$ is an open cover of A, so has a finite subcover $\{V_{a_i} \cap A\}_{i=1}^n$. Now, let $U = \cap_{i=1}^n U_{a_i}$, a neighborhood of x. Prove that $U \subset X - A$ (draw a picture!), which proves that $X - A$ is open.

53. (4/28) Prove that the only connected nonempty subsets of $\mathbb{Q}$ are its one-point subsets.

54. (4/28) Let X be a topological space and let $S \subseteq X$. Prove that if S is connected, then $\overline{S}$ is connected.

55. (4/28) Let X be an infinite set with the cofinite topology. Prove that X is connected.

56. (4/28) Suppose A and B are connected subsets of a space X. Prove that if $A \cap B$ is nonempty, then $A \cup B$ is connected.

57. (4/28) Let $X = C([0,1])$, the set of continuous functions from $[0,1]$ to $\mathbb{R}$. Define $\rho(f,g) = \max_{x\in[0,1]}\{(1-x^2)|f(x)-g(x)|\}$. Verify the triangle inequality. Hint: If $F(x) \leq G(x)$ for all $x \in [0,1]$, then $F(x) \leq \max_{x\in[0,1]}\{G(x)\}$ for all $x \in [0,1]$, and therefore $\max_{x\in[0,1]}\{F(x)\} \leq \max_{x\in[0,1]}\{G(x)\}$.

58. (4/28) Let (X,d) be a metric space and let A be a subset of X.
    (a) Use continuity of d to prove that if A is compact, then A is closed. Hint: If A is not closed, there is a limit point z of A that is not contained in A. Consider the function $f\colon A \to \mathbb{R}$ defined by $f(x) = 1/d(x,z)$.
    (b) Prove that X is Hausdorff, and apply Problem 52 to prove if A is compact, then A is closed.

59. (4/28) Let $f\colon X \to Y$ be continuous. Suppose $\{\, x_n \,\}$ is a sequence in X that converges to x. Prove that $\{\, f(x_n) \,\}$ converges to $f(x)$.

60. (4/28) Let X be a Hausdorff space. Prove that limits in X are unique. That is, if $\{\, x_n \,\}$ is a sequence in X and $x_n \to x$ and $x_n \to y$, then $x = y$.

61. Let $\{\, z_n = (x_n, y_n) \,\}$ be a sequence in $X \times Y$. Prove that $\{z_n\} \to (x,y)$ if and only if $\{x_n\} \to x$ and $\{y_n\} \to y$. Hint: For one direction, you can use an earlier problem applied to the projection functions.