**Math 4853 homework**

23. (3/1) Let $X$ be a set and let $\mathcal{B}$ be a collection of subsets of $X$. Define $\mathcal{U} = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U\}$. Prove that $U \in \mathcal{U}$ if and only if $U$ is a union of elements of $\mathcal{B}$.

24. (3/1) Let $X$ be a set and let $\mathcal{B}$ be a collection of subsets of $X$. We say that $\mathcal{B}$ is a *basis* if it satisfies:

    B1. $X = \cup_{B \in \mathcal{B}} B$.

    B2. $\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2, \exists B \in \mathcal{B}, x \in B \subseteq B_1 \cap B_2$.

    Define $\mathcal{U} = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U\}$. Prove that if $\mathcal{B}$ is a basis, then $\mathcal{U}$ is a topology on $X$.

25. (3/1) Let $X$ be a topological space and let $\mathcal{B}$ be a basis for the topology on $X$. Let $A \subset X$, and define $\mathcal{B}_A = \{B \cap A \mid B \in \mathcal{B}\}$.

    a. Prove that $\mathcal{B}_A$ is a basis.

    b. Prove that $\mathcal{B}_A$ generates the subspace topology on $A$. [Let $\mathcal{U}_A$ be the topology generated by $\mathcal{B}_A$, and let $\mathcal{U}$ be the subspace topology. Show that $U \in \mathcal{U}$ if and only if $U \in \mathcal{U}_A$. For the if direction, start with $U \in \mathcal{U}_A$, for each $a \in U$ choose $B_a$ with $a \in B_a \in U$, so that $U = \cup_{a \in U} B_a$, write each $B_a = B'_a \cap A$ with $B'_a \in \mathcal{B}$, and consider $V = \cup_{a \in U} B'_a$.]

26. (3/12) Take as known the fact that a composition of bijections is a bijection (prove this if it is not already clear to you).
    (a) Show that if $X$ and $Y$ are countable sets, then there is a bijection from $X$ to $Y$.
    (b) Let $X$ be a countable set and suppose there is a bijection from $Y$ to $X$. Show that $Y$ is also countable.

27. (3/12) State a theorem that the following argument proves: For each $x \in X$, $\Phi^{-1}(\{x\})$ is a nonempty subset of $\mathbb{N}$, so has a minimal element; define $\phi(x)$ to be the minimal element of $\Phi^{-1}(\{x\})$. If $x_1 \neq x_2$, then $\Phi^{-1}(\{x_1\}) \cap \Phi^{-1}(\{x_2\})$ is empty, so $\phi(x_1) \neq \phi(x_2)$. So $\phi$ is a bijection from $X$ to a subset $A$ of $\mathbb{N}$. Since $A$ must be countable, $X$ is also countable.

28. (3/12) Prove that the product $P = \{0,2\} \times \{0,2\} \times \{0,2\} \times \cdots = \{(a_1, a_2, a_3, \ldots) \mid a_i \in \{0,2\}\}$ is uncountable. Prove that the subset $A \subset P$ defined by $A = \{(a_1, a_2, \ldots) \in P \mid \exists N, \forall n > N, a_n = 0\}$ is countable.