Math 4853 homework

15. (2/12) Let $S(x,\epsilon) \subseteq \mathbb{R}^2$ be the open square of side 2ϵ centered at x. That is, $S((x_1,x_2),\epsilon) = \{(z_1,z_2) \mid |z_1 - x_1| < \epsilon \text{ and } |z_2 - x_2| < \epsilon\}$.

    (a) Prove that $S(x,\epsilon)$ is open. (To figure out a δ with $B(y,\delta) \subseteq S(x,\epsilon)$, draw a picture. The argument uses the fact that each $|z_i - x_i| \leq \| z - x \|$.)

    (b) Generalize to $\mathbb{R}^n$ by defining the open n-dimensional cube $S(x,\epsilon)$ in $\mathbb{R}^n$ and proving that it is open.

16. (2/12) (a) Prove that if $x \in \mathbb{R}^n$ and $\epsilon > 0$, then $S(x,\epsilon/\sqrt{n}) \subset B(x,\epsilon)$.

    (b) Let $U \subseteq \mathbb{R}^n$. Prove that U is open if and only if U is a union of open n-dimensional cubes.

17. Not to be turned in, but this task is a great way to prepare for next week's test: Write an exam over the material we have had in the course up until now. You will need to go back over what we have done and think about the major topics, ideas, and techniques. Think of some different kinds of questions such as giving important definitions, arguments that were steps in more complicated proofs we did, proofs for examples similar to examples we did in class, giving examples satisfying certain conditions, variations on homework problems. Try to focus on the conceptually more important matters rather than minutiae, and to cover a broad range of ideas and techniques. And it should be something that a student can reasonably be expected to complete in 50 minutes. If you want, exchange copies and try each other's tests. By the way, inexperienced test writers usually produce exams that are too long and/or too difficult.

18. (3/1) Verify that if U is open in the standard topology on $\mathbb{R}$, then it is open in the lower limit topology on $\mathbb{R}$.

19. (3/1) Verify that the cofinite topology on $\mathbb{R}$ is a topology. You will probably want to make use of DeMorgan's identities: If $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ are subsets of a set X, then $X - \cup_{\alpha\in\mathcal{A}} A_\alpha = \cap_{\alpha\in\mathcal{A}}(X - A_\alpha)$ and $X - \cap_{\alpha\in\mathcal{A}} A_\alpha = \cup_{\alpha\in\mathcal{A}}(X - A_\alpha)$.

20. (3/1) Define $\mathcal{A} = \{U \subseteq \mathbb{R} \,|\, U \text{ is finite}\} \cup \{\mathbb{R}\}$. Verify that $\mathcal{A}$ satisfies two of the three properties to be a topology on $\mathbb{R}$, but not the other one.

21. (3/1) Define $\mathcal{A} = \{U \subseteq \mathbb{R} \,|\, \forall x \in U, \exists a, b \in \mathbb{R}, \text{ either } x \in [a,b) \subseteq U \text{ or } x \in (a,b] \subseteq U\}$. Verify that $\mathcal{A}$ satisfies two of the three properties to be a topology on $\mathbb{R}$, but not the other one.

22. (3/1) Let $X = \mathbb{Q}$, the set of rational numbers, with the subspace topology as a subset of $\mathbb{R}$. Prove that if U is any nonempty open subset of $\mathbb{Q}$, then there exist nonempty open subsets U_1 and U_2 such that $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Notice that this appears not to be true for the standard topology on $\mathbb{R}$ (and indeed it is not true).