# Math 4853 homework

9. (2/12) For $1 \leq k \leq n$, let $\pi_k \colon \mathbb{R}^n \to \mathbb{R}$ be the projection function defined by $\pi_k(r_1, \ldots, r_n) = r_k$. Let $f \colon \mathbb{R}^m \to \mathbb{R}^n$ be a function. Define $f_k \colon \mathbb{R}^m \to \mathbb{R}$ by $f_k = \pi_k \circ f$, so that $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$.

   (a) Prove that $\pi_k$ is continuous. Deduce that if $f$ is continuous, then each $f_k$ is continuous.

   (b) Prove that if each $f_k$ is continuous, then $f$ is continuous. Hint: For each $k$, there exists $\delta_k$ so that $\| x - x_0 \| < \delta_k$ implies $|f_k(x) - f_k(x_0)| < \dfrac{\epsilon}{\sqrt{n}}$. Put $\delta = \min_{1 \leq k \leq n}\{\delta_k\}$.

10. (2/12) For a set $X$ we define $X \times X$ to be the set of ordered pairs of elements of $X$, that is, $X \times X = \{(a, b) \mid a, b \in X\}$. A *metric* on $X$ is a function $d \colon X \times X \to \mathbb{R}$ satisfying

    1. $d(a, b) \geq 0$ for all $a, b \in X$, and $d(a, b) = 0$ if and only if $a = b$.
    2. $d(a, b) = d(b, a)$ for all $a, b \in X$.
    3. $d(a, b) \leq d(a, c) + d(c, b)$ for all $a, b, c \in X$.

    (A metric is a function with the properties that we expect "distance" to have. For example, putting $d(x, y) = \| x - y \|$ defines a metric on $\mathbb{R}^n$.) If $X$ is a set with a metric $d$, then for $x \in X$ and $\epsilon > 0$, we define the *open ball of radius $\epsilon$ centered at $x$* to be $B(x, \epsilon) = \{z \in X \mid d(z, x) < \epsilon\}$. Prove the following:

    (i) For all $\epsilon > 0$, $a \in B(a, \epsilon)$.

    (ii) If $z \in B(x, \epsilon)$, then $\exists \delta > 0, B(z, \delta) \subseteq B(x, \epsilon)$.

    (iii) If a subset $W$ of $X$ is a union of open balls, then $\forall x \in W, \exists \epsilon > 0, B(x, \epsilon) \subseteq W$.

11. (no need to turn in, but ask about it in class if you have difficulty) For each of the following subsets of $\mathbb{R}$, determine whether the set is open, and whether its complement is open: $\{r \mid r > 0\}$, $\{r \mid r \text{ is not an integer }\}$, $\{r \mid r \text{ is rational}\}$.

12. (no need to turn in, but ask about it in class if you have difficulty) For each of the following subsets of $\mathbb{R}^2$, determine whether the set is open, and whether its complement is open: $\{(r_1, r_2) \mid r_1 > 0\}$, $\{(r_1, r_2) \mid r_1 \text{ is not an integer}\}$, $\{(r_1, 0) \mid r_1 \text{ is not an integer}\}$, $\{(r_1, r_2) \mid r_2 = \sin(r_1)\}$.