Math 4853 homework

1. (due 2/3) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$. Prove (directly from the definition) that f is continuous.

2. (2/3) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined as follows.

$$f(x) = \begin{cases} (q-1)/q & x \text{ is rational and } x = \pm\dfrac{p}{q} \text{ in lowest terms with } p \geq 0 \text{ and } q > 0 \\ 1 & x \text{ is irrational} \end{cases}$$

Prove (directly from the definition) that if x is irrational, then f is continuous at x.

3. (2/3) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined as follows.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

[where $\mathbb{Q}$ denotes the set of rational numbers] Use proof by contradiction to prove that f is not continuous at any x_0.

4. (2/3) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined as follows.

$$f(x) = \begin{cases} 1/q & x \text{ is rational and } x = p/q \text{ in lowest terms with } q > 0 \\ 0 & x \text{ is irrational} \end{cases}$$

Use proof by contradiction to prove that if x is rational, then f is not continuous at x.

5. (2/12) Prove directly from the definition of continuity that if $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuous at x_0, and $g(x_0) \neq 0$, then the quotient function f/g is continuous at x_0.

6. (2/12) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions. Prove that the composite function $g \circ f$ is continuous.

7. (2/12) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuous at x_0, and $g(x_0) \neq 0$, then the quotient function f/g is continuous at x_0 as follows: First prove that the reciprocal function defined by $k(x) = 1/x$ is continuous, then apply the facts that composites and products of continuous functions are continuous.

8. (2/12) Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}^k$ be continuous functions. Prove that the composite function $g \circ f\colon \mathbb{R}^m \to \mathbb{R}^k$ is continuous.