Math 4853 homework

1. (due 2/3) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be the function defined by \( f(x) = x^2 \). Prove (directly from the definition) that \( f \) is continuous.

2. (2/3) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be the function defined as follows.
   \[
   f(x) = \begin{cases} 
   (q - 1)/q & \text{if } x \text{ is rational and } x = \pm \frac{p}{q} \text{ in lowest terms with } p \geq 0 \text{ and } q > 0 \\
   1 & \text{if } x \text{ is irrational}
   \end{cases}
   \]
   Prove (directly from the definition) that if \( x \) is irrational, then \( f \) is continuous at \( x \).

3. (2/3) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be the function defined as follows.
   \[
   f(x) = \begin{cases} 
   1 & x \in \mathbb{Q} \\
   0 & x \notin \mathbb{Q}
   \end{cases}
   \]
   [where \( \mathbb{Q} \) denotes the set of rational numbers] Use proof by contradiction to prove that \( f \) is not continuous at any \( x_0 \).

4. (2/3) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be the function defined as follows.
   \[
   f(x) = \begin{cases} 
   1/q & x \text{ is rational and } x = p/q \text{ in lowest terms with } q > 0 \\
   0 & x \text{ is irrational}
   \end{cases}
   \]
   Use proof by contradiction to prove that if \( x \) is rational, then \( f \) is not continuous at \( x \).

5. (2/12) Prove directly from the definition of continuity that if \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) are continuous at \( x_0 \), and \( g(x_0) \neq 0 \), then the quotient function \( f/g \) is continuous at \( x_0 \).

6. (2/12) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) be continuous functions. Prove that the composite function \( g \circ f \) is continuous.

7. (2/12) Prove that if \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) are continuous at \( x_0 \), and \( g(x_0) \neq 0 \), then the quotient function \( f/g \) is continuous at \( x_0 \) as follows: First prove that the reciprocal function defined by \( k(x) = 1/x \) is continuous, then apply the facts that composites and products of continuous functions are continuous.

8. (2/12) Let \( f : \mathbb{R}^m \rightarrow \mathbb{R}^n \) and \( g : \mathbb{R}^n \rightarrow \mathbb{R}^k \) be continuous functions. Prove that the composite function \( g \circ f : \mathbb{R}^m \rightarrow \mathbb{R}^k \) is continuous.