Math 4853 homework

61. Let $\{ z_n = (x_n, y_n) \}$ be a sequence in $X \times Y$. Prove that $\{z_n\} \to (x, y)$ if and only if $\{x_n\} \to x$ and $\{y_n\} \to y$. Hint: For one direction, you can use an earlier problem applied to the projection functions.

Suppose first that $\{z_n\} \to (x, y)$. By problem 59, $\{\pi_X(z_n)\} \to \pi_X((x, y))$, that is, $\{x_n\} \to x$, and similarly $\{y_n\} \to y$. Conversely, assume that $\{x_n\} \to x$ and $\{y_n\} \to y$. Let W be a neighborhood of (x, y), and choose a basic open set $U \times V$ such that $(x, y) \in U \times V \subset W$. Since $x \in U$ and $y \in V$, there exist N_1 and N_2 such that if $n \geq N_1$ then $x_n \in U$, and if $n \geq N_2$ then $y_n \in V$. So if $n \geq \max\{N_1, N_2\}$, $(x_n, y_n) \in U \times V \subset W$.

62. Prove that every uncountable subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$. (Let A be an uncountable subset of $\mathbb{R}$, and for $n \in \mathbb{Z}$ put $A_n = A \cap [n, n + 1]$.)

Suppose that A is an uncountable subset of $\mathbb{R}$. For $n \in \mathbb{Z}$ put $A_n = A \cap [n, n + 1]$, so that $A = \cup_{n\in\mathbb{Z}} A_n$. If every A_n were finite, then A would be a countable union of finite sets, so would be countable. So some A_n, say A_N, is infinite. Since $A_N \subseteq [N, N + 1]$, which is compact, A_N has a limit point x_0 in $[N, N + 1]$. It is also a limit point of A in $\mathbb{R}$, since if U is any neighborhood of x_0 in $\mathbb{R}$, then $U \cap [N, N + 1]$ is a neighborhood of x_0 in $[N, N + 1]$, so contains a point of A_N other than x_0.

63. Let $\{x_n\}$ be a sequence in a metric space X. Prove that if $x_n \to x$, then $\{x_n\}$ is Cauchy.

Given $\epsilon > 0$, choose N so that if $n > N$, then $d(x_n, x) < \epsilon/2$. If $m, n > N$, then $d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \epsilon/2 + \epsilon/2 = \epsilon$.

64. Give $\mathbb{R}^k$ the metric $d(x, y) = \| x - y \|$. Let $\{z_n\}$ be a sequence of points in $\mathbb{R}^k$, written in coordinates as $z_n = (z_n^1, z_n^2, \ldots, z_n^k)$. Prove that $\{z_n\}$ is Cauchy if and only if each $\{z_n^i\}$ is a Cauchy sequence in $(\mathbb{R}, |x - y|)$.

Assume that $\{z_n\}$ is Cauchy. Given $\epsilon > 0$, choose N so that if $m, n > N$, then $\| z_m - z_n \| < \epsilon$. For this N and for each $1 \leq i \leq n$, we have $|z_m^i - z_n^i| = \sqrt{(z_m^i - z_n^i)^2} \leq \sqrt{\sum_{j=1}^n (z_m^j - z_n^j)^2} = \| z_m - z_n \| < \epsilon$.

Conversely, assume that each $\{z_n^i\}$ is Cauchy, and let $\epsilon > 0$ be given. For each i, there exists N_i such that if $m, n > N_i$, then $|z_m^i - z_n^i| < \epsilon/\sqrt{n}$. Let $N = \max N_i$. For $m, n > N$, we have $\| z_m - z_n \| = \sqrt{\sum_{i=1}^n (z_m^i - z_n^i)^2} < \sqrt{\sum_{i=1}^n (\epsilon/\sqrt{n})^2} = \sqrt{\sum_{i=1}^n \epsilon^2/n} = \sqrt{\epsilon^2} = \epsilon$.

65. Let $\{f_n\}$ be a sequence of functions in $C([0,1],\mathbb{R}^k)$ (the set of continuous functions from $[0,1]$ to $\mathbb{R}^k$. Prove that if $\{f_n\} \to f$ uniformly, then $\{f_n\} \to f$ pointwise.

Fix $x_0 \in [0,1]$, and let $\epsilon > 0$. Since $\{f_n\} \to f$ uniformly, there exists N so that if $n \geq N$, then for every $x \in [0,1]$, $\| f_n(x) - f(x) \| < \epsilon$. In particular, if $n \geq N$, then $\| f_n(x_0) - f(x_0) \| < \epsilon$. Therefore $\{f(x_0)\} \to f(x_0)$.

66. Let $f_n\colon [0,1] \to \mathbb{R}$ be $f_n(x) = x^n$, and let $f\colon [0,1] \to \mathbb{R}$ be defined by $f(x) = 0$ if $x < 1$ and $f(1) = 1$. Using the definitions, prove that $f_n \to f$ pointwise but not uniformly.

For pointwise convergence, suppose first that $0 \leq x_0 < 1$. Then by calculus, $\{x_0^n\} \to 0 = f(x_0)$. For $x_0 = 1$, $\{x_0^n\} = \{1\} \to 1 = f(x_0)$.

Suppose for contradiction that $\{x^n\} \to f$ uniformly. Then there exists N so that if $n \geq N$, then for all $x \in [0,1]$, $|x^n - f(x)| < 1/2$. Fix $n_0 > N$, and put $z_n = 1 - 1/n$. Then $\{z_n\} \to 1$. Since the function x^{n_0} is continuous, $\{z_n^{n_0}\} \to 1^{n_0} = 1$. Therefore there exists n_1 such that $z_{n_1}^{n_0} > 1/2$, so $|z_{n_1}^{n_0} - f(z_{n_1})| = |z_{n_1}^{n_0}| > 1/2$, a contradiction.