Math 4853 homework

61. Let \( \{z_n = (x_n, y_n)\} \) be a sequence in \( X \times Y \). Prove that \( \{z_n\} \to (x, y) \) if and only if \( \{x_n\} \to x \) and \( \{y_n\} \to y \). Hint: For one direction, you can use an earlier problem applied to the projection functions.

Suppose first that \( \{z_n\} \to (x, y) \). By problem 59, \( \{\pi_X(z_n)\} \to \pi_X((x, y)) \), that is, \( \{x_n\} \to x \), and similarly \( \{y_n\} \to y \). Conversely, assume that \( \{x_n\} \to x \) and \( \{y_n\} \to y \). Let \( W \) be a neighborhood of \( (x, y) \), and choose a basic open set \( U \times V \) such that \( (x, y) \in U \times V \subseteq W \).

Since \( x \in U \) and \( y \in V \), there exist \( N_1 \) and \( N_2 \) such that if \( n \geq N_1 \) then \( x_n \in U \), and if \( n \geq N_2 \) then \( y_n \in V \). So if \( n \geq \max\{N_1, N_2\} \), \( (x_n, y_n) \in U \times V \subseteq W \).

62. Prove that every uncountable subset of \( \mathbb{R} \) has a limit point in \( \mathbb{R} \). (Let \( A \) be an uncountable subset of \( \mathbb{R} \), and for \( n \in \mathbb{Z} \) put \( A_n = A \cap [n, n+1] \).)

Suppose that \( A \) is an uncountable subset of \( \mathbb{R} \). For \( n \in \mathbb{Z} \) put \( A_n = A \cap [n, n+1] \), so that \( A = \bigcup_{n \in \mathbb{Z}} A_n \). If every \( A_n \) were finite, then \( A \) would be a countable union of finite sets, so would be countable. So some \( A_n \), say \( A_N \), is infinite. Since \( A_N \subseteq [N, N+1] \), which is compact, \( A_N \) has a limit point \( x_0 \) in \([N, N+1]\). It is also a limit point of \( A \) in \( \mathbb{R} \), since if \( U \) is any neighborhood of \( x_0 \) in \( \mathbb{R} \), then \( U \cap [N, N+1] \) is a neighborhood of \( x_0 \) in \([N, N+1]\), so contains a point of \( A_N \) other than \( x_0 \).

63. Let \( \{x_n\} \) be a sequence in a metric space \( X \). Prove that if \( x_n \to x \), then \( \{x_n\} \) is Cauchy.

Given \( \epsilon > 0 \), choose \( N \) so that if \( n > N \), then \( d(x_n, x) < \epsilon/2 \). If \( m, n > N \), then \( d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \epsilon/2 + \epsilon/2 = \epsilon \).

64. Give \( \mathbb{R}^k \) the metric \( d(x, y) = \|x - y\| \). Let \( \{z_n\} \) be a sequence of points in \( \mathbb{R}^k \), written in coordinates as \( z_n = (z_n^1, z_n^2, \ldots, z_n^k) \). Prove that \( \{z_n\} \) is Cauchy if and only if each \( \{z_n^i\} \) is a Cauchy sequence in \( \mathbb{R}, \|x - y\| \).

Assume that \( \{z_n\} \) is Cauchy. Given \( \epsilon > 0 \), choose \( N \) so that if \( m, n > N \), then \( \|z_m - z_n\| < \epsilon \). For this \( N \) and for each \( 1 \leq i \leq n \), we have \( |z_m^i - z_n^i| = \sqrt{(z_m^i - z_n^i)^2} \leq \sqrt{\sum_{j=1}^{n}(z_m^j - z_n^j)^2} = \|z_m - z_n\| < \epsilon \).

Conversely, assume that each \( \{z_n^i\} \) is Cauchy, and let \( \epsilon > 0 \) be given. For each \( i \), there exists \( N_i \) such that if \( m, n > N_i \), then \( |z_m^i - z_n^i| < \epsilon/\sqrt{n} \). Let \( N = \max N_i \). For \( m, n > N \), we have \( \|z_m - z_n\| = \sqrt{\sum_{i=1}^{n}(z_m^i - z_n^i)^2} < \sqrt{\sum_{i=1}^{n}(\epsilon/\sqrt{n})^2} = \sqrt{\sum_{i=1}^{n} \epsilon^2/n} = \sqrt{\epsilon^2} = \epsilon \).
65. Let \( \{ f_n \} \) be a sequence of functions in \( C([0,1], \mathbb{R}^k) \) (the set of continuous functions from \([0,1]\) to \(\mathbb{R}^k\)). Prove that if \( \{ f_n \} \to f \) uniformly, then \( \{ f_n \} \to f \) pointwise.

Fix \( x_0 \in [0,1] \), and let \( \epsilon > 0 \). Since \( \{ f_n \} \to f \) uniformly, there exists \( N \) so that if \( n \geq N \), then for every \( x \in [0,1] \), \( \| f_n(x) - f(x) \| < \epsilon \). In particular, if \( n \geq N \), then \( \| f_n(x_0) - f(x_0) \| < \epsilon \). Therefore \( \{ f(x_0) \} \to f(x_0) \).

66. Let \( f_n : [0,1] \to \mathbb{R} \) be \( f_n(x) = x^n \), and let \( f : [0,1] \to \mathbb{R} \) be defined by \( f(x) = 0 \) if \( x < 1 \) and \( f(1) = 1 \). Using the definitions, prove that \( f_n \to f \) pointwise but not uniformly.

For pointwise convergence, suppose first that \( 0 \leq x_0 < 1 \). Then by calculus, \( \{ x_0^n \} \to 0 = f(x_0) \). For \( x_0 = 1 \), \( \{ x_0^n \} = \{ 1 \} \to 1 = f(x_0) \).

Suppose for contradiction that \( \{ x^n \} \to f \) uniformly. Then there exists \( N \) so that if \( n \geq N \), then for all \( x \in [0,1] \), \( |x^n - f(x)| < 1/2 \). Fix \( n_0 > N \), and put \( z_n = 1 - 1/n \). Then \( \{ z_n \} \to 1 \). Since the function \( x^{n_0} \) is continuous, \( \{ z^{n_0}_n \} \to 1^{n_0} = 1 \). Therefore there exists \( n_1 \) such that \( z^{n_0}_{n_1} > 1/2 \), so \( |z^{n_0}_{n_1} - f(z_{n_1})| = |z^{n_0}_{n_1}| > 1/2 \), a contradiction.