Math 4853 homework

43. (a) Let $B \subseteq A \subseteq X$. Prove that $B$ is closed in the subspace topology on $A$ if and only if there exists a closed subset $C \subseteq X$ such that $B = C \cap A$.

(b) Prove that if $A$ is a closed subset of $X$ and $B$ is a closed subset of $Y$, then $A \times B$ is a closed subset of $X \times Y$. Hint: Find a simple description of $X \times Y - A \times B$.

(c) Let $f: X \to Y$ be a function. Prove that $f$ is continuous if and only if for every closed subset $C \subseteq Y$, the inverse image $f^{-1}(C)$ is closed in $X$.

(a) Assume that $B$ is closed in $A$. Then $A - B$ is open in $A$, so there exists $V$ open in $X$ such that $V \cap A = A - B$. Since $V$ is open in $X$, $C = X - V$ is closed in $X$, and $C \cap A = (X - V) \cap A = A - V \cap A = B$. Conversely, assume that there exists $C$ closed in $X$ such that $C \cap A = B$. Since $C$ is closed, $X - C$ is open, and $(X - C) \cap A = A - (C \cap A) = A - B$, so $A - B$ is open in $A$ and therefore $B$ is closed in $A$.

(b) Observe that $X \times Y - A \times B = ((X - A) \times Y) \cup (X \times (Y - B))$ [since $(x, y) \notin A \times B \iff (x \notin A$ or $y \notin B) \iff x \in X - A$ or $y \in Y - B \iff (x, y) \in (X - A) \times Y$ or $(x, y) \in X \times (Y - B)$]. Since $A$ is closed, $X - A$ is open in $X$ and similarly $Y - B$ is open in $Y$, so $((X - A) \times Y) \cup (X \times (Y - B))$ is a union of two basic open sets and consequently is open.

(c) Assume that $f$ is continuous, and let $C$ be closed in $Y$. Then $Y - C$ is open, so $f^{-1}(Y - C) = X - f^{-1}(C)$ is open, and therefore $f^{-1}(C)$ is closed. Conversely, assume that $f^{-1}(C)$ is closed for every closed subset $C$ of $Y$. Let $U$ be open in $Y$. Then $Y - U$ is closed, so $f^{-1}(Y - U) = X - f^{-1}(U)$ is closed, so $f^{-1}(U)$ is open.

44. Let $S \subseteq X$.

(a) Prove that $x \in \overline{S}$ if and only if every neighborhood of $x$ contains a point of $S$.

(b) Prove that $S$ is closed if and only if $S = \overline{S}$.

(c) Prove that $\overline{S} = \cap\{A \subseteq X \mid A$ is closed and $S \subseteq A\}$.

(d) Let $f: X \to Y$ be continuous. Prove that $f(\overline{S}) \subseteq \overline{f(S)}$.

(e) Give an example of a continuous surjective function $f: X \to Y$ and a subset $S \subseteq X$ such that $f(S) \neq f(\overline{S})$.

(a) Assume that $x \in \overline{S}$. Let $U$ be an open neighborhood of $x$. If $x \in S$, then $U$ contains the point $x$ of $S$. If $x \in S'$, then $U - \{x\}$ contains a point of $S$. In either case, $U$ contains a point of $S$. Conversely, assume that every neighborhood of $x$ contains a point of $S$. If $x \in S$, then $x \in S \cup S' = \overline{S}$. If $x \notin S$, then $x \in S'$ since every neighborhood of $x$ contains a point of $S$, which cannot be $x$ since $x \notin S$. In either case, $x \in \overline{S}$.

(b) Assume that $S$ is closed. Since $S \subseteq S$, Proposition 2 says that $\overline{S} \subseteq S$. By definition, $S \subseteq S \cup S' = \overline{S}$. Therefore $S = \overline{S}$. Conversely, assume that $S = \overline{S}$. By Proposition 1, $\overline{S}$ and therefore $S$ are closed.

(c) If $A$ is any closed set with $S \subseteq A$, then by Proposition 2, $\overline{S} \subseteq A$. Therefore $\overline{S} \subseteq \cap\{A \subseteq X \mid A$ is closed and $S \subseteq A\}$. On the other hand, $\overline{S}$ is a closed set that contains $S$, so is among the sets in the collection being intersected in the expression $\cap\{A \subseteq X \mid A$ is closed and $S \subseteq A\} \subseteq \overline{S}$. 

(d) Let \( x \in \mathcal{S} \). Let \( U \) be any neighborhood of \( f(x) \). Then \( f^{-1}(U) \) is open and \( x \in f^{-1}(U) \). Since \( x \in \mathcal{S} \), \( f^{-1}(U) \) must contain a point \( s \) of \( S \). Then, \( f(s) \in f(S) \cap U \). We have shown that every neighborhood of \( f(x) \) contains a point of \( f(S) \), so by part (a), \( f(x) \in \overline{f(S)} \).

(e) Example 1: Let \( X = (-2, -1) \cup [1, 2], \ Y = [1, 4] \), and \( f : X \to Y \) be defined by \( f(x) = x^2 \). Then \( f((-2, -1)) = f((2, -1)) = (1, 4) \neq (1, 4) = f((-2, -1)) \).

Example 2: Let \( X = [0, 2\pi] \), \( Y = S^1 \), and \( f : X \to Y \) be \( f(t) = (\cos(t), \sin(t)) \). Let \( S = [\pi, 2\pi] \), which is closed in \( X \). The image \( f(S) \) is not closed in \( Y \), so \( f(S) \neq f(\overline{S}) \).

Example 3: Let \( X = \mathbb{R}^2 \) and \( Y = \mathbb{R} \), and let \( \pi : X \to Y \) be projection to the first coordinate. Let \( S = \{(x, y) \mid x \neq 0 \text{ and } y = 1/x\} \). Then \( S \) is closed in \( X \), but \( \pi(S) = \mathbb{R} - \{0\} \) is not closed in \( Y \), so \( f(S) = f(S) \neq f(\overline{S}) \).

45. Let \( X \) be \( \mathbb{R} \) with the lower-limit topology, and let \( A \) be the subspace \([0, 1]\) of \( X \). Give an example of a continuous unbounded function from \( A \) to \( \mathbb{R} \).

Define \( f : X \to \mathbb{R} \) by \( f(x) = 1/(1-x) \) if \( x < 1 \) and \( f(x) = 0 \) if \( x \geq 1 \). This is continuous, since for every \( x_0 \), the limit from the right \( \lim_{x \to x_0^+} f(x) \) equals \( f(x_0) \). Let \( f_A : A \to \mathbb{R} \) be the restriction of \( f \). Then \( f_A \) is continuous, since it is the restriction of a continuous function to a subspace, and \( f_A \) is unbounded.

46. Let \( X = \{1/n \mid n \in \mathbb{N}\} \cup \{0\} \), a subspace of \( \mathbb{R} \). Prove that every continuous function \( f : X \to \mathbb{R} \) is bounded, by considering the open set \( V = (f(0) - 1, f(0) + 1) \).

Let \( V = f^{-1}((f(0) - 1, f(0) + 1)) \). Since \( f \) is continuous, this is an open neighborhood of 0 in \( X \). Since it is open in \( X \), it must contain \((-\epsilon, \epsilon) \cap X \) for some \( \epsilon > 0 \). Choose \( N \) with \( 1/N < \epsilon \), and let \( M = \max\{f(0)+1, f(1), f(1/2), \ldots, f(1/N)\} \). We claim that \( M \) is an upper bound for \( f(X) \). Let \( y \in f(X) \). If \( y = f(0) \), then \( f(0) < f(0) + 1 \leq M \). If \( y = f(1/n) \) for \( n \leq N \), then \( f(1/n) \in \{f(0) + 1, f(1), f(1/2), \ldots, f(1/N)\} \) so \( f(1/n) \leq M \). Finally, if \( y = f(1/n) \) for \( n \geq N \), then \( 1/n \in (-\epsilon, \epsilon) \cap X \in f^{-1}(V) \), so \( f(1/n) \in V \) and therefore \( f(1/n) < f(0) + 1 \leq M \). In any case, \( y \leq M \). Similarly, \( \min\{f(0) - 1, f(1), f(1/2), \ldots, f(1/N)\} \) is a lower bound for \( f(X) \).

47. Let \( X \) be a topological space. Prove that if \( X \) is compact, then every continuous function \( f : X \to \mathbb{R} \) is bounded. Use the open cover \( \{V_n\}_{n \in \mathbb{N}} \) of \( \mathbb{R} \), where \( V_n = (-n, n) \).

For \( n \in \mathbb{N} \), define \( U_n = f^{-1}(V_n) \), an open subset of \( X \). We have \( X = f^{-1}(\mathbb{R}) = f^{-1}(\bigcup_{n \in \mathbb{N}} V_n) = \bigcup_{n \in \mathbb{N}} f^{-1}(V_n) = \bigcup_{n \in \mathbb{N}} U_n \), so \( \{U_n\}_{n \in \mathbb{N}} \) is an open cover of \( X \). Since \( X \) is compact, this has a finite subcover, say \( \{U_{n_i}\}_{i=1}^k \). So \( f(X) \subseteq \bigcup_{i=1}^k V_{n_i} = V_N \) where \( N = \max\{n_i\} \). That is, \( f(X) \subseteq (-N, N) \) so \( f \) is bounded.

48. (4/14) For any set \( X \), the cofinite topology on \( X \) is the topology in which a set is open if and only if it is either empty or has finite complement. Prove that any set \( X \) with the cofinite topology is compact.
Let $X$ have the cofinite topology. If $X$ is empty, it is compact, so we may assume that $X$ is nonempty. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $X$. Some $U_{\alpha_0}$ must be nonempty. Write $U_{\alpha_0} = X - F$, where $F$ is finite, say $F = \{x_1, x_2, \ldots, x_k\}$. For each $i$ with $1 \leq i \leq k$, choose some $U_{\alpha_i}$ with $x_i \in U_{\alpha_i}$. Then, $\{U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ is a finite subcover of $\{U_\alpha\}_{\alpha \in A}$. For let $x$ be any element of $X$. If $x \in F$, say $x = x_i$, then $x \in U_{\alpha_i}$. If $x \notin F$, then $x \in U_{\alpha_0}$.

49. Prove that if $A$ is a compact subset of $\mathbb{R}$, then $A$ is bounded (i.e., $A$ lies in some interval $[-M, M]$).

For $n \geq 1$, let $U_n = A \cap (-n, n)$, and open subset of $A$. Since $A$ is compact, there exists a finite subcollection of the $U_n$ such that $A = \bigcup_{n=1}^{k} U_{n_i} = \bigcup_{i=1}^{k} (A \cap (-n_i, n_i)) = A \cap (\bigcup_{i=1}^{k} (-n_i, n_i)) = A \cap (-N, N)$, where $N = \max\{n_1, \ldots, n_k\}$. So $A$ lies in some finite interval and therefore is bounded.

50. Prove that if $A$ is a compact subset of $\mathbb{R}$, then $A$ is closed. (Hint: It seems easiest to argue the contrapositive: if $A$ is not closed then it is not compact. If $A$ is not closed, then $A \neq \overline{A} = A \cup A'$, so there is some limit point $x_0$ of $A$ that is not contained in $A$. Then...

We will prove the contrapositive. Assume that $A$ is not closed. Then $A \neq \overline{A} = A \cup A'$, so there is some limit point $x_0$ of $A$ that is not contained in $A$. Consider the continuous function $f: \mathbb{R} - \{x_0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x - x_0}$, and let $g: A \to \mathbb{R}$ be the restriction of $f$ to $A$. We will show that $g$ is unbounded.

Suppose for contradiction that $g$ is bounded. Then there exists $M$ so that $g(A) \subseteq [-M, M]$. We may choose $M > 0$. That is, $a \in A$ implies $|g(a)| \leq M$. This says $\frac{1}{|a - x_0|} \leq M$, so $|a - x_0| \geq \frac{1}{M}$. Therefore there is no point of $A$ in the interval $(x_0 - \frac{1}{M}, x_0 + \frac{1}{M})$, contradicting the fact that $x_0$ is a limit point of $A$.

[One can obtain the contradiction directly from the definition as follows: For $n \in \mathbb{N}$, let $U_n = (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \cap A$. Since $x_0 \notin A$, the $U_n$ form an open cover of $A$, and since $x_0 \in A'$, there is no finite subcover.]