Math 4853 homework

43. (a) Let $B \subseteq A \subseteq X$. Prove that B is closed in the subspace topology on A if and only if there exists a closed subset $C \subseteq X$ such that $B = C \cap A$.
(b) Prove that if A is a closed subset of X and B is a closed subset of Y, then $A \times B$ is a closed subset of $X \times Y$. Hint: Find a simple description of $X \times Y - A \times B$.
(c) Let $f\colon X \to Y$ be a function. Prove that f is continuous if and only if for every closed subset $C \subseteq Y$, the inverse image $f^{-1}(C)$ is closed in X.

(a) Assume that B is closed in A. Then $A - B$ is open in A, so there exists V open in X such that $V \cap A = A - B$. Since V is open in X, $C = X - V$ is closed in X, and $C \cap A = (X - V) \cap A = A - V \cap A = B$. Conversely, assume that there exists C closed in X such that $C \cap A = B$. Since C is closed, $X - C$ is open, and $(X - C) \cap A = A - (C \cap A) = A - B$, so $A - B$ is open in A and therefore B is closed in A.

(b) Observe that $X \times Y - A \times B = ((X - A) \times Y) \cup (X \times (Y - B))$ [since $(x, y) \notin A \times B \Leftrightarrow (x \notin A \text{ or } y \notin B) \Leftrightarrow x \in X - A \text{ or } y \in Y - B \Leftrightarrow (x, y) \in (X - A) \times Y \text{ or } (x, y) \in X \times (Y - B)$]. Since A is closed, $X - A$ is open in X and similarly $Y - B$ is open in Y, so $((X - A) \times Y) \cup (X \times (Y - B))$ is a union of two basic open sets and consequently is open.

(c) Assume that f is continuous, and let C be closed in Y. Then $Y - C$ is open, so $f^{-1}(Y - C) = X - f^{-1}(C)$ is open, and therefore $f^{-1}(C)$ is closed. Conversely, assume that $f^{-1}(C)$ is closed for every closed subset C of Y. Let U be open in Y. Then $Y - U$ is closed, so $f^{-1}(Y - U) = X - f^{-1}(U)$ is closed, so $f^{-1}(U)$ is open.

44. Let $S \subset X$.
(a) Prove that $x \in \overline{S}$ if and only if every neighborhood of x contains a point of S.
(b) Prove that S is closed if and only if $S = \overline{S}$.
(c) Prove that $\overline{S} = \cap\{\, A \subseteq X \mid \text{A is closed and } S \subseteq A \,\}$.
(d) Let $f\colon X \to Y$ be continuous. Prove that $f(\overline{S}) \subseteq \overline{f(S)}$.
(e) Give an example of a continuous surjective function $f\colon X \to Y$ and a subset $S \subset X$ such that $f(\overline{S}) \neq \overline{f(S)}$.

(a) Assume that $x \in \overline{S}$. Let U be an open neighborhood of x. If $x \in S$, then U contains the point x of S. If $x \in S'$, then $U - \{x\}$ contains a point of S. In either case, U contains a point of S. Conversely, assume that every neighborhood of x contains a point of S. If $x \in S$, then $x \in S \cup S' = \overline{S}$. If $x \notin S$, then $x \in S'$ since every neighborhood of x contains a point of S, which cannot be x since $x \notin S$. In either case, $x \in \overline{S}$.

(b) Assume that S is closed. Since $S \subset S$, Proposition 2 says that $\overline{S} \subseteq S$. By definition, $S \subseteq S \cup S' = \overline{S}$. Therefore $S = \overline{S}$. Conversely, assume that $S = \overline{S}$. By Propostiion 1, $\overline{S}$ and therefore S are closed.

(c) If A is any closed set with $S \subseteq A$, then by Proposition 2, $\overline{S} \subseteq A$. Therefore $\overline{S} \subseteq \cap\{\, A \subseteq X \mid \text{A is closed and } S \subseteq A \,\}$. On the other hand, $\overline{S}$ is a closed set that contains S, so is among the sets in the collection being intersected in the expression $\cap\{\, A \subseteq X \mid \text{A is closed and } S \subseteq A \,\}$. Therefore $\cap\{\, A \subseteq X \mid \text{A is closed and } S \subseteq A \,\} \subseteq \overline{S}$.

(d) Let $x \in \overline{S}$. Let U be any neighborhood of $f(x)$. Then $f^{-1}(U)$ is open and $x \in f^{-1}(U)$. Since $x \in \overline{S}$, $f^{-1}(U)$ must contain a point s of S. Then, $f(s) \in f(S) \cap U$. We have shown that every neighborhood of $f(x)$ contains a point of $f(S)$, so by part (a), $f(x) \in \overline{f(S)}$.

(e) Example 1: Let $X = (-2,-1) \cup [1,2]$, $Y = [1,4]$, and $f\colon X \to Y$ be define by $f(x) = x^2$. Then $f(\overline{(-2,-1)}) = f((-2,-1)) = (1,4) \neq \overline{(1,4)} = f((-2,-1))$.

Example 2: Let $X = [0, 2\pi)$, $Y = S^1$, and $f\colon X \to Y$ be $f(t) = (\cos(t), \sin(t))$. Let $S = [\pi, 2\pi)$, which is closed in X. The image $f(S)$ is not closed in Y, so $f(\overline{S}) = f(S) \neq \overline{f(S)}$.

Example 3: Let $X = \mathbb{R}^2$ and $Y = \mathbb{R}$, and let $\pi\colon X \to Y$ be projection to the first coordinate. Let $S = \{(x,y) \mid x \neq 0 \text{ and } y = 1/x\}$. Then S is closed in X, but $\pi(S) = \mathbb{R} - \{0\}$ is not closed in Y, so $f(\overline{S}) = f(S) \neq \overline{f(S)}$.

45. Let X be $\mathbb{R}$ with the lower-limit topology, and let A be the subspace $[0,1]$ of X. Give an example of a continuous unbounded function from A to $\mathbb{R}$.

Define $f : X \to \mathbb{R}$ by $f(x) = 1/(1-x)$ if $x < 1$ and $f(x) = 0$ if $x \geq 1$. This is continuous, since for every x_0, the limit from the right $\lim_{x \to x_0^+}$ equals $f(x_0)$. Let $f_A\colon A \to \mathbb{R}$ be the restriction of f. Then f_A is continuous, since it is the restriction of a continuous function to a subspace, and f_A is unbounded.

46. Let $X = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$, a subspace of $\mathbb{R}$. Prove that every continuous function $f\colon X \to \mathbb{R}$ is bounded, by considering the open set $V = (f(0) - 1, f(0) + 1)$.

Let $V = f^{-1}((f(0) - 1, f(0) + 1))$. Since f is continuous, this is an open neighborhood of 0 in X. Since it is open in X, it must contain $(-\epsilon, \epsilon) \cap X$ for some $\epsilon > 0$. Choose N with $1/N < \epsilon$, and let $M = \max\{f(0)+1, f(1), f(1/2), \ldots, f(1/N)\}$. We claim that M is an upper bound for $f(X)$. Let $y \in f(X)$. If $y = f(0)$, then $f(0) < f(0) + 1 \leq M$. If $y = f(1/n)$ for $n < N$, then $f(1/n) \in \{f(0) + 1, f(1), f(1/2), \ldots, f(1/N)\}$ so $f(1/n) \leq M$. Finally, if $y = f(1/n)$ for $n \geq N$, then $1/n \in (-\epsilon, \epsilon) \cap X \subseteq f^{-1}(V)$, so $f(1/n) \in V$ and therefore $f(1/n) < f(0) + 1 \leq M$. In any case, $y \leq M$. Similarly, $\min\{f(0) - 1, f(1), f(1/2), \ldots, f(1/N)\}$ is a lower bound for $f(X)$.

47. Let X be a topological space. Prove that if X is compact, then every continuous function $f\colon X \to \mathbb{R}$ is bounded. Use the open cover $\{V_n\}_{n \in \mathbb{N}}$ of $\mathbb{R}$, where $V_n = (-n, n)$.

For $n \in \mathbb{N}$, define $U_n = f^{-1}(V_n)$, nd open subset of X. We have $X = f^{-1}(\mathbb{R}) = f^{-1}(\cup_{n \in \mathbb{N}} V_n) = \cup_{n \in \mathbb{N}} f^{-1}(V_n) = \cup_{n \in \mathbb{N}} U_n$, so $\{U_n\}_{n \in \mathbb{N}}$ is an open cover of X. Since X is compact, this has a finite subcover, say $\{U_{n_i}\}_{i=1}^k$. So $f(X) \subseteq \cup_{i=1}^k V_{n_i} = V_N$ where $N = \max\{n_i\}$. That is, $f(X) \subseteq (-N, N)$ so f is bounded.

48. (4/14) For any set X, the *cofinite topology* on X is the topology in which a set is open if and only if it is either empty or has finite complement. Prove that any set X with the cofinite topology is compact.

Let X have the cofinite topology. If X is empty, it is compact, so we may assume that X is nonempty. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of X. Some U_{α_0} must be nonempty. Write $U_{\alpha_0} = X - F$, where F is finite, say $F = \{x_1, x_2, \ldots, x_k\}$. For each i with $1 \leq i \leq k$, choose some U_{α_i} with $x_i \in U_{\alpha_i}$. Then, $\{U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ is a finite subcover of $\{U_\alpha\}_{\alpha \in \mathcal{A}}$. For let x be any element of X. If $x \in F$, say $x = x_i$, then $x \in U_{\alpha_i}$. If $x \notin F$, then $x \in U_{\alpha_0}$.

49. Prove that if A is a compact subset of $\mathbb{R}$, then A is bounded (i. e. A lies in some interval $[-M, M]$).

For $n \geq 1$, let $U_n = A \cap (-n, n)$, and open subset of A. Since A is compact, there exists a finite subcollection of the U_n such that $A = \cup_{i=1}^k U_{n_i} = \cup_{i=1}^k (A \cap (-n_i, n_i)) = A \cap (\cup_{i=1}^k (-n_i, n_i)) = A \cap (-N, N)$, where $N = \max\{n_1, \ldots n_k\}$. So A lies in some finite interval and therefore is bounded.

50. Prove that if A is a compact subset of $\mathbb{R}$, then A is closed. (Hint: It seems easiest to argue the contrapositive: if A is not closed then it is not compact. If A is not closed, then $A \neq \overline{A} = A \cup A'$, so there is some limit point x_0 of A that is not contained in A. Then...)

We will prove the contrapositive. Assume that A is not closed. Then $A \neq \overline{A} = A \cup A'$, so there is some limit point x_0 of A that is not contained in A. Consider the continuous function $f\colon \mathbb{R} - \{x_0\} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x - x_0}$, and let $g\colon A \to \mathbb{R}$ be the restriction of f to A. We will show that g is unbounded,

Suppose for contradiction that g is bounded. Then there exists M so that $g(A) \subseteq [-M, M]$. We may choose $M > 0$. That is, $a \in A$ implies $|g(a)| \leq M$. This says $\dfrac{1}{|a - x_0|} \leq M$, so $|a - x_0| \geq \dfrac{1}{M}$. Therefore there is no point of A in the interval $(x_0 - \frac{1}{M}, x_0 + \frac{1}{M})$, contradicting the fact that x_0 is a limit point of A.

[One can obtain the contradiction directly from the definition as follows: For $n \in \mathbb{N}$, let $U_n = (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \cap A$. Since $x_0 \notin A$, the U_n form an open cover of A, and since $x_0 \in A'$, there is no finite subcover.]