Math 4853 homework

35. Let $f\colon X \to Y$ be a function between topological spaces. Let Z be a subset of Y such that $f(X) \subseteq Z$, and define $g\colon X \to Z$ by $g(x) = f(x)$. We say that g is obtained from f by restriction of the codomain. Assuming, of course, that Z has the subspace topology as a subspace of Y, prove that f is continuous if and only if g is. (Moral of the story: It's OK to be careless about codomains.)

Assume first that g is comtinous. Then f is the composition of g followed by the inclusion of Z into Y, so f is also continuous.

Assume now that f is continuous, and let U be open in Z. Then $U = Z \cap V$ for some open subset in Y. We have $x \in g^{-1}(U)$ if and only if $g(x) \in U$ if and only if $f(x) \in U$ if and only if $x \in f^{-1}(V)$, so $g^{-1}(U) = f^{-1}(V)$. Since f is continuous, $f^{-1}(V)$ is open, and therefore $g^{-1}(U)$ is open.

36. Prove that the continuous bijection $f\colon [0, 2\pi) \to S^1$ defined by $f(t) = (\cos(t), \sin(t))$ is not a homeomorphism.

We must show that $f^{-1}\colon S^1 \to [0, 2\pi)$ is not continuous. Let $U = (\infty, \pi) \cap [0, 2\pi)$, an open subset of $[0, 2\pi)$ [note: many other choices of U are possible in the argument]. Now $(f^{-1})^{-1}(U) = f(U)$ is the set of points in S^1 with y-coordinate positive, together with the point $(1, 0)$. But if V is any open set in $\mathbb{R}^2$ containing $(1, 0)$, then V contains an open ball around $(1, 0)$ and hence $V \cap S^1$ contains points of S^1 below the x-axis. So $f(U)$ is not open in S^1 and therefore f^{-1} is not continuous.

37. (a) Let $\mathcal{C}$ be the cofinite topology on $\mathbb{R}$. Prove that if $f^{-1}(\{r\})$ finite for every $r \in \mathbb{R}$, then $f\colon (\mathbb{R}, \mathcal{C}) \to (\mathbb{R}, \mathcal{C})$ is continuous. [You will need to use the fact that if $g\colon X \to Y$ is a function and $S \subseteq Y$, then $g^{-1}(Y - S) = X - g^{-1}(S)$, which you should check if it is not clear to you.]
(b) Examine the converse of part (a).

Assume that $f^{-1}(\{r\})$ finite for every $r \in \mathbb{R}$. Let U be open in the cofinite toplogy. If U is empty, then so is $f^{-1}(U)$, so we may assume that U is nonempty, say $U = \mathbb{R} - F$ for a finite set F. For each $r \in F$, $f^{-1}(\{r\})$ is a finite set F_r, so $f^{-1}(F) = \cup_{r\in F} F_r$ is finite. We have $f^{-1}(U) = f^{-1}(\mathbb{R} - F) = \mathbb{R} - f^{-1}(F)$, so $f^{-1}(U)$ is open.

For the converse, suppose that f is continuous, and consider the sets open sets $\mathbb{R} - \{r\}$. Each $f^{-1}(\mathbb{R} - \{r\})$ is open, that is $\mathbb{R} - f^{-1}(\{r\})$ is open. If for some r this is empty, then $f^{-1}(\{r\}) = \mathbb{R}$ so f is constant. Otherwise, $f^{-1}(\{r\})$ is finite for every r. So the converse of the previous part is not quite true. A correct statement is: $f\colon (\mathbb{R}, \mathcal{C}) \to (\mathbb{R}, \mathcal{C})$ is continuous if and only if either f is constant or $\forall r \in \mathbb{R}$, $f^{-1}(\{r\})$ is finite.

38. (not to turn in) Work through the following step-by-step argument proving that the following are equivalent for a bijection $\phi\colon X \to Y$ between topological spaces:

(i) ϕ is a homeomorphism.

(ii) U is open in X if and only if $\phi(U)$ is open in Y.

First, we observe that if $\phi\colon X \to Y$ is a bijection, and $A \subseteq X$, then $(\phi^{-1})^{-1}(A) = \phi(A)$. For we have $x \in (\phi^{-1})^{-1}(U) \Leftrightarrow \phi^{-1}(x) \in U \Leftrightarrow x = \phi(\phi^{-1}(x)) \in \phi(U)$.

Assume (i). Suppose first that U is open in X. Then $\phi(U) = (\phi^{-1})^{-1}(U)$ is open in Y, since ϕ^{-1} is continuous. Suppose now that $\phi(U)$ is open in Y. Then $U = \phi^{-1}(\phi(U))$ is open in X, since ϕ is continuous.

Now assume (ii). Let V be open in Y. Since $V = \phi(\phi^{-1}(V))$, (ii) implies that $\phi^{-1}(V)$ is open in X, so ϕ is continuous. Let U be open in X. Then $(\phi^{-1})^{-1}(U) = \phi(U)$, which is open by (ii). Therefore ϕ^{-1} is continuous.

39. In $\mathbb{R}$, let $\mathcal{S} = \{(a,\infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$. Prove that $\mathcal{S}$ is a sub-basis that generates the standard topology. Find a similar sub-basis for the lower-limit topology.

We first note that $\cap_{i=1}^{m}(a_i,\infty) = (a,\infty)$ where $a = \max\{a_i\}$, and similarly $\cap_{i=1}^{n}(-\infty, b_j) = (-\infty, b)$ where $b = \min\{b_j\}$. So any finite intersection $S_1 \cap \cdots \cap S_n$ of elements of $\mathcal{S}$ can be expressed as $(a,\infty) \cap (-\infty, b) = (a, b)$. That is, the basis $\mathcal{B} = \{S_1 \cap \cdots \cap S_n \mid S_i \in \mathcal{S}\}$ equals $\{(a,b) \mid a, b \in \mathbb{R}\}$, one of the known bases for the standard topology on $\mathbb{R}$.

40. Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases for the topologies of two spaces X and Y. Prove that $\{B_1 \times B_2 \mid B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_Y\}$ is a basis for the product topology on $X \times Y$.

It suffices to verify the hypotheses of the Basis Recognition Theorem. Each element of $\mathcal{B}_X$ is open in X and similarly for $\mathcal{B}_Y$, so each $B_1 \times B_2$ is open in $X \times Y$. Let W be an open subset of $X \times Y$, and let $(x_0, y_0) \in W$. Then there exists a basis open set $U \times V$ with $(x_0, y_0) \in U \times V \subset W$. Now, $x_0 \in U$, an open subset of X, so there exists a basic open set $B_1 \in \mathcal{B}_X$ with $x_0 \in B_1 \subseteq U$. Similarly there exists $B_2 \in \mathcal{B}_Y$ with $y_0 \in B_2 \subseteq V$. So we have $(x_0, y_0) \in B_1 \times B_2 \subseteq U \times V \subset W$.

41. (4/2) Let $f\colon X \to Y$ be a function. Recall that the *graph* of f is the subset $\Gamma_f \subset X \times Y$ defined by $\Gamma_f = \{(x, y) \mid f(x) = y\}$. Assume that X is a topological space and that Y is a *Hausdorff* topological space, which means that if y_1 and y_2 are any two points of Y, then there are *disjoint* open sets U_1 and U_2 in Y with $y_1 \in U_1$ and $y_2 \in U_2$. Prove that if f is continuous, then the complement $X \times Y - \Gamma_f$ of Γ_f is an open subset of $X \times Y$. Hint: Write $W = X \times Y - \Gamma_f$. It suffices to show that if $(x_0, y_0) \in W$, then there is a basic open set W' with $(x_0, y_0) \in W' \subseteq W$. Since $(x_0, y_0) \in W$, $f(x_0) \neq y_0$ and therefore there are disjoint open sets V_1 and V_2 in Y with $f(x_0) \in V_1$ and $y_0 \in V_2$. Now examine $W' = f^{-1}(V_1) \times V_2$ (draw a picture!).

Write $W = X \times Y - \Gamma_f$. It suffices to show that if $(x_0, y_0) \in W$, then there is a basic open set W' with $(x_0, y_0) \in W' \subseteq W$. It suffices to show that if $(x_0, y_0) \in W$, then there is a basic open set W' with $(x_0, y_0) \in W_{(x_0,y_0)} \subseteq W$. Since $(x_0, y_0) \in W$, $f(x_0) \neq y_0$ and therefore there are disjoint open sets V_1 and V_2 in Y with $f(x_0) \in V_1$ and $y_0 \in V_2$. Since f is continuous, $f^{-1}(V_1)$ is open in X. Let $W' = f^{-1}(V_1) \times V_2$, a basic open subset of $X \times Y$. We have $(x_0, y_0) \in W'$, since $f(x_0) \in V_1$ and $y_0 \in V_2$. To show that $W' \subseteq W$, we will show that $W' \cap \Gamma_f = \emptyset$. Suppose that $(x, y) \in W'$. Then $x \in f^{-1}(V_1)$ so $f(x) \in V_1$ and therefore $f(x) \notin V_2$. On the other hand, $y \in V_2$, so $y \neq f(x)$. Therefore $(x, y) \notin \Gamma_f$.

42. Let $X \times Y$ be a product of topological spaces. Prove that for each $x_0 \in X$, the subspace $\{x_0\} \times Y$ is homeomorphic to Y. Show, in fact, that the restriction π of the projection function $\pi_Y \colon X \times Y \to Y$ is a homeomorphism. Hint: Let $j \colon Y \to \{x_0\} \times Y$ be defined by $j(y) = (x_0, y)$. Observe that j is an inverse function to π, hence π is bijective. Give a simple reason why π is continuous, and apply a theorem to show that j is continuous. (Of course, the same kind of arguments would show that each $X \times \{y_0\}$ is homeomorphic to X.)

Let $\pi = \pi_Y|_{\{x_0\} \times Y} \colon \{x_0\} \times Y \to Y$. The function $j \colon Y \to \{x_0\} \times Y$ defined by $j(y) = (x_0, y)$ is an inverse function to π, since $j \circ \pi(x_0, y) = j(y) = (x_0, y)$ and $\pi \circ j(y) = \pi(x_0, y) = y$, so π is bijective. Since π is the restriction of a continuous function, it is continuous. The coordinate function $\pi_X \circ j$ is the constant function sending Y to $x_0 \in X$, so is continuous, while $\pi_Y \circ j$ is the identity function on Y. Since the coordinate functions of j are continuous, j is continuous. Therefore π and j are homeomorphisms.