Math 4853 homework

26. Take as known the fact that a composition of bijections is a bijection (prove this if it is not already clear to you).
(a) Show that if X and Y are countable sets, then there is a bijection from X to Y.
(b) Let X be a countable set and suppose there is a bijection from Y to X. Show that Y is also countable.

Assume that X and Y are countable, and let $\phi_X : X \to \mathbb{N}$ and $\phi_Y : Y \to \mathbb{N}$ be bijections. Then $\phi_Y^{-1} \circ \phi_X$ is a bijection from X to Y.

For (b), assume that X is countable and that $\phi: Y \to X$ is a bijection. Since X is countable, there exists a bijective function $\phi_X : X \to \mathbb{N}$. But then, $\phi_X \circ \phi$ is a bijective function from Y to X, so Y is also countable.

27. State a theorem that the following argument proves: For each $x \in X$, $\Phi^{-1}(\{x\})$ is a nonempty subset of $\mathbb{N}$, so has a minimal element; define $\phi(x)$ to be the minimal element of $\Phi^{-1}(\{x\})$. If $x_1 \neq x_2$, then $\Phi^{-1}(\{x_1\}) \cap \Phi^{-1}(\{x_2\})$ is empty, so $\phi(x_1) \neq \phi(x_2)$. So ϕ is a bijection from X to a subset A of $\mathbb{N}$. Since A must be countable, X is also countable.

It proves: If there exists a surjective function $\Phi: \mathbb{N} \to X$, then X is countable.

28. Prove that the product $P = \{0,2\} \times \{0,2\} \times \{0,2\} \times \cdots = \{(a_1, a_2, a_3, \ldots) \mid a_i \in \{0,2\}\}$ is uncountable. Prove that the subset $A \subset P$ defined by $A = \{(a_1, a_2, \ldots) \in P \mid \exists N, \forall n > N, a_n = 0\}$ is countable.

Suppose for contradiction that there is a bijection from P to $\mathbb{N}$, then its inverse is a bijection $\phi: \mathbb{N} \to P$. For each $n \in \mathbb{N}$, write $\phi(n) = (a_1, a_2, \ldots)$ and put $b_n = 2$ if $a_n = 0$ and $b_n = 0$ if $a_n = 2$. Let $b = (b_1, b_2, b_3, \ldots)$. Then b differs in the n^{th} coordinate from $\phi(n)$, so $b \neq \phi(n)$ for any n. This contradicts the surjectivity of ϕ.

For each N define $P_N = \{(a_1, a_2, \ldots) \mid a_n = 0 \text{ for all } n > N\}$. Then an element a of P is in A if and only if it lies in some P_N. Therefore $A = \cup_{N=1}^{\infty} P_N$ is a union of countably many countable sets, so A is countable.

29. Let X be a topological space. Suppose that $\mathcal{B}$ is a collection of subsets of X such that

(1) The sets in $\mathcal{B}$ are open in X.

(2) For every open set U in X and every $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

(a) Prove that $\mathcal{B}$ is a basis.

(b) Prove that the topology $\{V \subseteq X \mid \forall x \in V, \exists B \in \mathcal{B}, x \in B \subseteq V\}$ generated by $\mathcal{B}$ equals the topology of X.

For (a), let $x \in X$. By (2), there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq X$. Therefore $X = \cup_{x\in X}\{x\} \subseteq \cup_{x\in X} B_x \subseteq X$, so $X = \cup_{x\in X} B_x$. Now, let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. By (1), B_1 and B_2 are open, so $B_1 \cap B_2$ is open. By (2), there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

For (b), suppose first that U is open in X. For each $x \in U$, (2) says there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Therefore U is in the topology generated by $\mathcal{B}$.

Suppose now that V is in the topology generated by $\mathcal{B}$. By definition, for each $x \in V$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq V$. Therefore $V = \cup_{x\in V}\{x\} \subseteq \cup_{x\in V} B_x \subseteq V$, so $V = \cup_{x\in V} B_x$. By (1), each B_x is open in X, so V, being a union of open sets in X, is also open.

30. Use the previous problem to give a quick proof that if X is a topological space, A is a subset of X, and $\mathcal{B}$ is a basis for the topology of X, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on A.

Each B is open in X, so each $B \cap A$ is open in the subspace topology by definition. Now let $U \subset A$ be open in the subspace topology, and let $a \in U$. By definition, there exists V open in X such that $U = V \cap A$. Since $x \in U \subseteq V$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq V$. Therefore $x \in B \cap A \subseteq V \cap A = U$. By the Basis Recognition Theorem, $\{B \cap A \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on A.

31. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions between topological spaces.

(a) Check that for any subset $V \subseteq Z$, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$.

(b) Prove that if f and g are continuous, then $g \circ f$ is continuous.

(c) Give an example of non-continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ whose composition is continuous.

For (a), $x \in (g \circ f)^{-1}(V)$ if and only if $(g \circ f)(x) \in V$ if and only if $g(f(x)) \in V$ if and only if $f(x) \in g^{-1}(V)$ if and only if $x \in f^{-1}(g^{-1}(V))$.

For (b), suppose that f and g are continuous and W is open in Z. Since g is continuous, $g^{-1}(W)$ is open in Y. Since f is continuous, $f^{-1}(g^{-1}(W))$ is open in X. By (a), this says that $(g \circ f)^{-1}(W)$ is open in X, so $g \circ f$ is continuous.

For (c), define $f, g\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 1$ if $x < 0$ and $f(x) = 2$ if $x \geq 0$, and $g(x) = 0$ if $x < 0$ and $g(x) = 1$ if $x \geq 0$. Neither f nor g is continuous, but $g \circ f$ is constant and hence is continuous (and $f \circ f$, $f \circ g$, and $g \circ g$ are also constant).

Another example is $f(x) = -x$ if $x \in \mathbb{Q}$ and $f(x) = x$ if $x \notin Q$, $g(x) = x$ if $x \in \mathbb{Q}$ and $g(x) = -x$ if $x \notin Q$. Then, f and g are discontinuous, but all of $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ are continuous.

32. Let $f\colon X \to Y$ be a function between topological spaces, and let $\mathcal{B}$ be a basis for the topology on Y. Prove that if $f^{-1}(B)$ is open for every $B \in \mathcal{B}$, then f is continuous.

Assume that $f^{-1}(B)$ is open for every $B \in \mathcal{B}$. Let V be open in Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since $\mathcal{B}$ is a basis, there exists $B \in \mathcal{B}$ such that $f(x) \in B \subseteq V$. Therefore $x \in f^{-1}(B) \subseteq f^{-1}(V)$. By the usual argument, this shows that $f^{-1}(V)$ is a union of sets having the form $f^{-1}(B)$ with $B \in \mathcal{B}$. By assumption, every such $f^{-1}(B)$ is open in X, so $f^{-1}(V)$ is open in X. Therefore f is continuous.

33. Check any of the properties $f^{-1}(\cup A_\alpha) = \cup f^{-1}(A_\alpha)$, $f^{-1}(\cap A_\alpha) = \cap f^{-1}(A_\alpha)$, and $f(\cup A_\alpha) = \cup f(A_\alpha)$ that are not clear to you. Give a counterexample to $f(A \cap B) = f(A) \cap f(B)$.

One counterexample in $\mathbb{R}$ is $A = [0, 2\pi]$, $B = [4\pi, 6\pi]$, and $g(x) = \sin(x)$, so $g(A) \cap g(B) = [-1, 1] \cap [-1, 1] = [-1, 1]$, but $g(A \cap B) = g(\emptyset) = \emptyset$. After finding a counterexample, did you go on to find and prove the strongest general statement: $f(A \cap B) \subseteq f(A) \cap f(B)$?

34. Complete the proof that if $f\colon (\mathbb{R}, \mathcal{S}) \to (\mathbb{R}, \mathcal{L})$ is continuous, then f is constant by completing Case II (assume that $x_1 < x_2$ and $f(x_1) > f(x_2)$ and reach a contradiction. Do it analogously to our method for Case I but using the greatest lower bound property— every nonempty set of real numbers that has a lower bound has a greatest lower bound. (There is also a simple trick for deducing Case II from Case I, but the point here is to practice this type of argument. If you want, try to also find the simple trick method.)

Suppose for contradiction that $x_1 < x_2$ and $f(x_1) > f(x_2)$. Let $S = \{x \in [x_1, x_2] \mid f(x) < f(x_1)\}$. Since S contains x_2, it is nonempty and bounded below, so it has a greatest lower bound L.

Suppose first that $f(L) \geq f(x_1)$. Then $L \in f^{-1}([f(x_1), \infty))$, an open set in $(\mathbb{R}, \mathcal{S})$, so there exists $\epsilon > 0$ such that $(L - \epsilon, L + \epsilon) \subseteq f^{-1}([f(x_1), \infty))$, so $(L - \epsilon, L + \epsilon) \cap S = \emptyset$. Also $(-\infty, L) \cap S = \emptyset$, since L is a lower bound for S. So $L + \epsilon$ is a lower bound for S, contradicting the fact that L is the greatest lower bound.

Suppose now that $f(L) < f(x_1)$. Then $L \in f^{-1}((-\infty, f(x_1)))$, an open set in $(\mathbb{R}, \mathcal{S})$, so there exists $\epsilon > 0$ such that $(L - \epsilon, L + \epsilon) \subseteq f^{-1}((-\infty, f(x_1)))$. But then, $(L - \epsilon, L + \epsilon) \subseteq S$ and hence $L - \epsilon/2 \in S$, contradicting the fact that L is a lower bound for S.

Here is the simple trick: Suppose for contradiction that $x_1 < x_2$ and $f(x_1) > f(x_2)$. Let $g\colon (\mathbb{R}, \mathcal{S}) \to (\mathbb{R}, \mathcal{S})$ be $g(x) = -x$, a continuous function. Since f is continuous, so is $k = f \circ g\colon (\mathbb{R}, \mathcal{S}) \to (\mathbb{R}, \mathcal{L})$. Now $-x_2 < -x_1$, and $k(-x_2) = f(x_2) < f(x_1) = k(-x_1)$. By Case I, this is impossible.