Math 4853 homework solutions (version of February 23, 2010)

5. Prove directly from the definition of continuity that if $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuous at x_0, and $g(x_0) \neq 0$, then the quotient function f/g is continuous at x_0.

Fix x_0 and let $\epsilon > 0$ be given.

Since f is continuous at x_0 and $g(x_0) \neq 0$, there exists $\delta_1 > 0$ so that if $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < \epsilon |g(x_0)|/4$.

Since g is continuous at x_0 and $g(x_0) \neq 0$, there exists $\delta_2 > 0$ so that if $|x - x_0| < \delta_2$, then $|g(x) - g(x_0)| < \epsilon |g(x_0)|^2/(4(|f(x_0)| + 1))$.

Since g is continuous at x_0 and $g(x_0) \neq 0$, there exists $\delta_3 > 0$ so that if $|x - x_0| < \delta_2$, then $|g(x) - g(x_0)| < |g(x_0)|/2$. Notice that the latter implies that $|g(x_0)| = |g(x_0) - g(x) + g(x)| \leq |g(x_0) - g(x)| + |g(x)| < |g(x_0)|/2 + |g(x)|$, so $|g(x)| > |g(x_0)|/2$.

For $|x - x_0| < \min\{\delta_1, \delta_2, \delta_3\}$, we have

$$
\begin{aligned}
|(f/g)(x) - (f/g)(x_0)| &= \left| \frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0} \right| = \left| \frac{f(x)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)} \right| \\
&= \left| \frac{(f(x) - f(x_0))g(x_0) + f(x_0)(g(x_0) - g(x))}{g(x)g(x_0)} \right| \\
&\leq \frac{|f(x) - f(x_0)|}{|g(x)g(x_0)|}\,|g(x_0)| + |f(x_0)|\,\frac{|g(x_0) - g(x)|}{|g(x)g(x_0)|} \\
&= \frac{|f(x) - f(x_0)|}{|g(x)|} + |f(x_0)|\,\frac{|g(x_0) - g(x)|}{|g(x)|\,|g(x_0)|} \\
&< \frac{|f(x) - f(x_0)|}{|g(x_0)|/2} + |f(x_0)|\,\frac{|g(x_0) - g(x)|}{|g(x_0)|^2/2} \\
&< \frac{\epsilon |g(x_0)|/4}{|g(x_0)|/2} + |f(x_0)|\,\frac{\epsilon |g(x_0)|^2/(4(|f(x_0)| + 1))}{|g(x_0)|^2/2} \\
&= \frac{\epsilon}{2} + \frac{|f(x_0)|}{|f(x_0)| + 1}\,\frac{\epsilon}{2} < \epsilon
\end{aligned}
$$

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions. Prove that the composite function $g \circ f$ is continuous.

Fix x_0 and let $\epsilon > 0$ be given. Since g is continuous at $f(x_0)$, there exists $\delta_1 > 0$ so that if $|z - f(x_0)| < \delta_1$, then $|g(z) - g(f(x_0))|$.

Since f is continuous at x_0, there exists $\delta > 0$ so that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \delta_1$.

If $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \delta_1$ and consequently $|g(f(x)) - g(f(x_0))| < \epsilon$.

7. Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuous at x_0, and $g(x_0) \neq 0$, then the quotient function f/g is continuous at x_0 as follows: First prove that the reciprocal function defined by $k(x) = 1/x$ is continuous, then apply the facts that composites and products of continuous functions are continuous.

Let $x_0 \neq 0$ be given. Notice that if $|x - x_0| < |x_0|/2$, then $|x_0| = |x_0 - x + x| \leq |x_0 - x| + |x| < |x_0|/2 + |x|$ and consequently $|x| > |x_0|/2$. Put $\delta = \epsilon |x_0|^2/2$. If $|x - x_0| < \delta$, then

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{x x_0}\right| = \frac{|x_0 - x|}{|x|\,|x_0|} < \frac{\epsilon |x_0|^2/2}{|x_0|^2/2} = \epsilon\ .$$

This shows that the reciprocal function is continuous.

Suppose that f and g are continuous. By Problem 6 (at all points where $g(x) \neq 0$), $k \circ g$ is continuous. Since a product of continuous functions is continuous, it follows that $f/g = f \cdot (k \circ g)$ continuous.

8. Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}^k$ be continuous functions. Prove that the composite function $g \circ f\colon \mathbb{R}^m \to \mathbb{R}^k$ is continuous.

(the proof is the same as the proof in Problem 6, except that norms must be used instead of absolute values).

9. For $1 \leq k \leq n$, let $\pi_k\colon \mathbb{R}^n \to \mathbb{R}$ be the projection function defined by $\pi_k(r_1, \ldots, r_n) = r_k$. Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a function. Define $f_k\colon \mathbb{R}^m \to \mathbb{R}$ by $f_k = \pi_k \circ f$, so that $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$.
   (a) Prove that π_k is continuous. Deduce that if f is continuous, then each f_k is continuous.
   (b) Prove that if each f_k is continuous, then f is continuous. Hint: For each k, there exists δ_k so that $\|\, x - x_0 \,\| < \delta_k$ implies $|f_k(x) - f_k(x_0)| < \dfrac{\epsilon}{\sqrt{n}}$. Put $\delta = \min_{1 \leq k \leq n}\{\delta_k\}$.

For (a), given $x \in \mathbb{R}^n$ and $\epsilon > 0$, put $\delta = \epsilon$. If $\|\, z - x \,\| < \delta$, then $|\pi_k(z) - \pi_k(x)| = |z_k - x_k| = \sqrt{(z_k - x_k)^2} \leq \sqrt{\sum_{i=1}^n (z_i - x_i)^2} = \|\, z - x \,\| < \epsilon$.

For (b), assume that each f_k is continuous. Let $x \in \mathbb{R}^n$ and for each k, choose δ_k such that if $\|\, z - x \,\| < \delta_k$ then $|f_k(z) - f_k(z)| < \epsilon/\sqrt{n}$. Then for $\|\, z - x \,\| < \min_{1 \leq k \leq n}\{\delta_k\}$, we have

$$\begin{aligned}\|\, f(z) - f(x) \,\| &= \left(\sum_{k=1}^n (f_k(z) - f_k(x))^2\right)^{1/2} = \left(\sum_{k=1}^n |f_k(z) - f_k(x)|^2\right)^{1/2} \\ &< \left(\sum_{k=1}^n (\epsilon/\sqrt{n})^2\right)^{1/2} = \left(\sum_{k=1}^n \epsilon^2/n\right)^{1/2} = \sqrt{\epsilon^2} = \epsilon\end{aligned}$$

10. For a set X we define $X \times X$ to be the set of ordered pairs of elements of X, that is, $X \times X = \{(a, b) \mid a, b \in X\}$. A *metric* on X is a function $d\colon X \times X \to \mathbb{R}$ satisfying
    1. $d(a, b) \geq 0$ for all $a, b \in X$, and $d(a, b) = 0$ if and only if $a = b$.
    2. $d(a, b) = d(b, a)$ for all $a, b \in X$.
    3. $d(a, b) \leq d(a, c) + d(c, b)$ for all $a, b, c \in X$.

    (A metric is a function with the properties that we expect "distance" to have. For example, putting $d(x, y) = \|\, x - y \,\|$ defines a metric on $\mathbb{R}^n$.) If X is a set with a metric d, then for $x \in X$ and $\epsilon > 0$, we define the *open ball of radius* ϵ *centered at* x to be $B(x, \epsilon) = \{z \in X \mid d(z, x) < \epsilon\}$. Prove the following:

(i) For all $\epsilon > 0$, $a \in B(a, \epsilon)$.
(ii) If $z \in B(x, \epsilon)$, then $\exists \delta > 0, B(z, \delta) \subseteq B(x, \epsilon)$.
(iii) If a subset W of X is a union of open balls, then $\forall x \in W, \exists \epsilon > 0, B(x, \epsilon) \subseteq W$.

For (i), $d(x, x) = 0 < \epsilon$ by property 1, so $x \in B(x, \epsilon)$.

For (ii), given $z \in B(x, \epsilon)$, put $\delta = \epsilon - d(z, x)$. Suppose that $y \in B(z, \delta)$. Using property 3, $d(y, x) \leq d(y, z) + d(z, x) < \delta + d(z, x) = \epsilon$, so $y \in B(x, \epsilon)$. Therefore $B(z, \delta) \subseteq B(x, \epsilon)$.

For (iii), suppose that $W = \cup_{\alpha \in \mathcal{A}} B(x_\alpha, \epsilon_\alpha)$ is a union of open balls. Let $x \in W$. Then $x \in B(x_\beta, \epsilon_\beta)$ for some $\beta \in \mathcal{A}$. By (ii), there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq B(x_\beta, \epsilon_\beta) \subseteq W$.

11. (no need to turn in, but ask about it in class if you have difficulty) For each of the following subsets of $\mathbb{R}$, determine whether the set is open, and whether its complement is open: $\{r \mid r > 0\}$, $\{r \mid r \text{ is not an integer}\}$, $\{r \mid r \text{ is rational}\}$.

$\{r \mid r > 0\}$ is open, its complement is not.

$\{r \mid r \text{ is not an integer}\}$ is open, its complement is not.

$\{r \mid r \text{ is rational}\}$, is not open, neither is its complement.

12. (no need to turn in, but ask about it in class if you have difficulty) For each of the following subsets of $\mathbb{R}^2$, determine whether the set is open, and whether its complement is open: $\{(r_1, r_2) \mid r_1 > 0\}$, $\{(r_1, r_2) \mid r_1 \text{ is not an integer}\}$, $\{(r_1, 0) \mid r_1 \text{ is not an integer}\}$, $\{(r_1, r_2) \mid r_2 = \sin(r_1)\}$.

$\{(r_1, r_2) \mid r_1 > 0\}$ is open, its complement is not.

$\{(r_1, r_2) \mid r_1 \text{ is not an integer}\}$ is open, its complement is not.

$\{(r_1, 0) \mid r_1 \text{ is not an integer}\}$ is not open, nor is its complement.

$\{(r_1, r_2) \mid r_2 = \sin(r_1)\}$ is not open, its complement is open.

13. Let $S(x, \epsilon) \subseteq \mathbb{R}^2$ be the open square of side 2ϵ centered at x. That is, $S((x_1, x_2), \epsilon) = \{(z_1, z_2) \mid \| z_1 - x_1 \| < \epsilon \text{ and } \| z_2 - x_2 \| < \epsilon\}$.
    (a) Prove that $S(x, \epsilon)$ is open. (To figure out a δ with $B(y, \delta) \subseteq S(x, \epsilon)$, draw a picture. The argument uses the fact that each $|z_i - x_i| \leq \| z - x \|$.)
    (b) Generalize to $\mathbb{R}^n$ by defining the open n-dimensional cube $S(x, \epsilon)$ in $\mathbb{R}^n$ and proving that it is open.

For (a), let $y \in S(x, \epsilon)$. Then $|y_1 - x_1| < \epsilon$ and $|y_2 - x_2| < \epsilon$, so $\delta = \min\{\epsilon - |y_1 - x_1|, \epsilon - |y_2 - x_2|\} > 0$. If $z \in B(y, \delta)$, then $|z_1 - x_1| \leq |z_1 - y_1| + |y_1 - x_1| < \| z - y \| + |y_1 - x_1| < \delta + |y_1 - x_1| \leq \epsilon - |y_1 - x_1| + |y_1 - x_1| = \epsilon$, and similarly $|z_x - x_2| < \epsilon$, so $z \in S(x, \epsilon)$.

For (b), let $y \in S(x, \epsilon)$. Then for $1 \leq i \leq n$, $|y_i - x_i| < \epsilon$. Therefore $\delta = \min_{1 \leq i \leq n}\{\epsilon - |y_i - x_i|\} > 0$. If $z \in B(y, \delta)$, then for each i, $|z_i - x_i| \leq |z_i - y_i| + |y_i - x_i| < \| z - y \| + |y_i - x_i| < \delta + |y_i - x_i| \leq \epsilon - |y_i - x_i| + |y_i - x_i| = \epsilon$, so $y \in S(x, \epsilon)$.

14. (2/12) (a) Prove that if $x \in \mathbb{R}^n$ and $\epsilon > 0$, then $S(x, \epsilon/\sqrt{n}) \subset B(x, \epsilon)$.
    (b) Let $U \subset \mathbb{R}^n$. Prove that U is open if and only if U is a union of open n-dimensional cubes.

For (a), let $z \in S(x, \epsilon)$. Then

$$\| z - x \| = \sqrt{\sum_{i=1}^{n}(z_i - x_i)^2} = \sqrt{\sum_{i=1}^{n}|z_i - x_i|^2} < \sqrt{\sum_{i=1}^{n}(\epsilon/\sqrt{n})^2} < \sqrt{n(\epsilon)^2/n} = \epsilon \ ,$$

so $z \in B(x, \epsilon)$.

For (b), suppose first that U is open, and let $x \in U$. Then there is an ϵ_x such that $B(x, \epsilon_x) \subset U$. By part (a), $S(x, \epsilon_x/\sqrt{n}) \subset B(x, \epsilon_x) \subset U$. Therefore $U = \cup_{x \in U}\{x\} \subseteq \cup_{x \in U} S(x, \epsilon_x/\sqrt{n}) \subset U$, so $U = \cup_{x \in U} S(x, \epsilon_x/\sqrt{n})$.

Conversely, suppose that $U = \cup_{\alpha \in \mathcal{A}} S(x_\alpha, \epsilon_\alpha)$. Let $x \in U$. Then $x \in S(x_\beta, \epsilon_\beta)$ for some $\beta \in \mathcal{A}$. By Problem 13, $S(x_\beta, \epsilon_\beta)$ is open, so there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq S(x_\beta, \epsilon_\beta) \subseteq U$. Therefore U is open.

[Or, you can give the conversely argument in words: Suppose that U is a union of open cubes. Each of the open cubes is open, by Problem 13, so is a union of open balls. Taking the union of all those open balls gives U.]

15. Let $S(x, \epsilon) \subseteq \mathbb{R}^2$ be the open square of side 2ϵ centered at x. That is, $S((x_1, x_2), \epsilon) = \{(z_1, z_2) \mid \| z_1 - x_1 \| < \epsilon \text{ and } \| z_2 - x_2 \| < \epsilon\}$.
    (a) Prove that $S(x, \epsilon)$ is open. (To figure out a δ with $B(y, \delta) \subseteq S(x, \epsilon)$, draw a picture. The argument uses the fact that each $|z_i - x_i| \leq \| z - x \|$.)
    (b) Generalize to $\mathbb{R}^n$ by defining the open n-dimensional cube $S(x, \epsilon)$ in $\mathbb{R}^n$ and proving that it is open.

For (a), let $y \in S(x, \epsilon)$. Then $|y_1 - x_1| < \epsilon$ and $|y_2 - x_2| < \epsilon$, so $\delta = \min\{\epsilon - |y_1 - x_1|, \epsilon - |y_2 - x_2|\} > 0$. If $z \in B(y, \delta)$, then $|z_1 - x_1| \leq |z_1 - y_1| + |y_1 - x_1| < \| z - y \| + |y_1 - x_1| < \delta + |y_1 - x_1| \leq \epsilon - |y_1 - x_1| + |y_1 - x_1| = \epsilon$, and similarly $|z_x - x_2| < \epsilon$, so $z \in S(x, \epsilon)$.

For (b), let $y \in S(x, \epsilon)$. Then for $1 \leq i \leq n$, $|y_i - x_i| < \epsilon$. Therefore $\delta = \min_{1 \leq i \leq n}\{\epsilon - |y_i - x_i|\} > 0$. If $z \in B(y, \delta)$, then for each i, $|z_i - x_i| \leq |z_i - y_i| + |y_i - x_i| < \| z - y \| + |y_i - x_i| < \delta + |y_i - x_i| \leq \epsilon - |y_i - x_i| + |y_i - x_i| = \epsilon$, so $y \in S(x, \epsilon)$.

16. (a) Prove that if $x \in \mathbb{R}^n$ and $\epsilon > 0$, then $S(x, \epsilon/\sqrt{n}) \subset B(x, \epsilon)$.
    (b) Let $U \subset \mathbb{R}^n$. Prove that U is open if and only if U is a union of open n-dimensional cubes.

For (a), let $z \in S(x, \epsilon)$. Then

$$\| z - x \| = \sqrt{\sum_{i=1}^{n}(z_i - x_i)^2} = \sqrt{\sum_{i=1}^{n}|z_i - x_i|^2} < \sqrt{\sum_{i=1}^{n}(\epsilon/\sqrt{n})^2} < \sqrt{n(\epsilon)^2/n} = \epsilon \ ,$$

so $z \in B(x, \epsilon)$.

For (b), suppose first that U is open, and let $x \in U$. Then there is an ϵ_x such that $B(x, \epsilon_x) \subset U$. By part (a), $S(x, \epsilon_x/\sqrt{n}) \subset B(x, \epsilon_x) \subset U$. Therefore $U = \cup_{x \in U}\{x\} \subseteq \cup_{x \in U} S(x, \epsilon_x/\sqrt{n}) \subset U$, so $U = \cup_{x \in U} S(x, \epsilon_x/\sqrt{n})$.

Conversely, suppose that $U = \cup_{\alpha \in \mathcal{A}} S(x_\alpha, \epsilon_\alpha)$. Let $x \in U$. Then $x \in S(x_\beta, \epsilon_\beta)$ for some $\beta \in \mathcal{A}$. By Problem 13, $S(x_\beta, \epsilon_\beta)$ is open, so there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq S(x_\beta, \epsilon_\beta) \subseteq U$. Therefore U is open.

[Or, you can give the conversely argument in words: Suppose that U is a union of open cubes. Each of the open cubes is open, by Problem 13, so is a union of open balls. Taking the union of all those open balls gives U.]

17. Not to be turned in, but this task is a great way to prepare for next week's test: Write an exam over the material we have had in the course up until now. You will need to go back over what we have done and think about the major topics, ideas, and techniques. Think of some different kinds of questions such as giving important definitions, arguments that were steps in more complicated proofs we did, proofs for examples similar to examples we did in class, giving examples satisfying certain conditions, variations on homework problems. Try to focus on the conceptually more important matters rather than minutiae, and to cover a broad range of ideas and techniques. And it should be something that a student can reasonably be expected to complete in 50 minutes. If you want, exchange copies and try each other's tests. By the way, inexperienced test writers usually produce exams that are too long and/or too difficult.