Math 4853 homework solutions (version of February 12, 2010)

1. Let \( f: \mathbb{R} \to \mathbb{R} \) be the function defined by \( f(x) = x^2 \). Prove (directly from the definition) that \( f \) is continuous.

Fix \( x_0 \), and suppose for now that \( x_0 \neq 0 \). Note first that if \( |x - x_0| < |x_0| \), then \( |x| = |x - x_0 + x_0| \leq |x - x_0| + |x_0| < |x_0| + |x_0| = 2|x_0| \). Now, given \( \epsilon > 0 \), put \( \delta = \min\{|x_0|, \epsilon/(3|x_0|)\} \); since \( x_0 \neq 0 \) and \( \delta > 0 \).

Then, if \( |x - x_0| < \delta \), we have

\[
|x^2 - x_0^2| = |x + x_0||x - x_0| < 3|x_0|\epsilon/(3|x_0|) = \epsilon .
\]

Suppose now that \( x_0 = 0 \). Given \( \epsilon > 0 \), put \( \delta = \sqrt{\epsilon} \). If \( |x - 0| < \delta \), then \( |x^2 - 0^2| = x^2 < (\sqrt{\epsilon})^2 = \epsilon \).

2. Let \( f: \mathbb{R} \to \mathbb{R} \) be the function defined as follows.

\[ f(x) = \begin{cases} 
(q - 1)/q & \text{x is rational and x = \pm \frac{p}{q} in lowest terms with } p \geq 0 \text{ and } q > 0 \\
1 & \text{x is irrational}
\end{cases} \]

Prove (directly from the definition) that if \( x \) is irrational, then \( f \) is continuous at \( x \).

Given \( \epsilon \), choose a positive integer \( N \) with \( 1/N < \epsilon \). Let \( S \) be the set of rational numbers in the interval \( (x - 1, x + 1) \) with the following property:

If \( r_i \) is written as \( p_i/q_i \) with \( p_i \) and \( q_i \) integers in lowest terms, with \( q_i > 0 \), then \( q_i \leq N \).

We note that \( S \) is nonempty, since there is at least one integer in the interval \( (x - 1, x + 1) \) for which the denominator is 1, and \( S \) is finite, since the interval \( (x - 1, x + 1) \) has finite length. So we can write \( S = \{r_1, \ldots, r_k\} \).

Each \( |r_i - x| > 0 \), since \( x \) is irrational, so the number \( S_{min} = \min\{|r_i - x|\}_{r_i \in S} \) is positive. Put \( \delta = \min\{S_{min}, 1\} \).

Case I: \( z \) is irrational

In this case \( |f(z) - f(x)| = |1 - 1| = 0 < \epsilon \).

Case II: \( z \) is rational

Since \( |z - x| < \delta \leq 1 \), \( z \in (x - \delta, x + \delta) \), but since \( |z - x| < \delta \), \( z \) cannot equal any \( r_i \). So \( z = \frac{p}{q} \) in lowest terms with \( q > N \). Therefore \( |f(z) - f(x)| = |(q - 1)/q - 1| = 1/q < 1/N < \epsilon \).

3. Let \( f: \mathbb{R} \to \mathbb{R} \) be the function defined as follows.

\[ f(x) = \begin{cases} 
1 & x \in \mathbb{Q} \\
0 & x \notin \mathbb{Q}
\end{cases} \]

[where \( \mathbb{Q} \) denotes the set of rational numbers] Use proof by contradiction to prove that \( f \) is not continuous at any \( x_0 \).
Fix $x_0$ and suppose for contradiction that $f$ is continuous at $x_0$. Then there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < 1$.

Case I: $x_0$ is rational.

Choose an irrational $x$ in the interval $(x_0 - \delta, x_0 + \delta)$. Then $|x - x_0| < \delta$, but $|f(x) - f(x_0)| = |0 - 1| = 1$, a contradiction.

Case II: $x_0$ is irrational.

Choose a rational $x$ in the interval $(x_0 - \delta, x_0 + \delta)$. Then $|x - x_0| < \delta$, but $|f(x) - f(x_0)| = |1 - 0| = 1$, a contradiction.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows.

\[
    f(x) = \begin{cases} 
        1/q & x \text{ is rational and } x = p/q \text{ in lowest terms with } q > 0 \\
        0 & x \text{ is irrational}
    \end{cases}
\]

Use proof by contradiction to prove that if $x$ is rational, then $f$ is not continuous at $x$.

Fix a rational number $x_0$ and suppose for contradiction that $f$ is continuous at $x_0$. Write $x_0 = p/q$ in lowest terms with $q > 0$. Then there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < 1/q$. Choose an irrational $x$ in the interval $(x_0 - \delta, x_0 + \delta)$. Then $|x - x_0| < \delta$, but $|f(x) - f(x_0)| = |0 - 1/q| = 1/q$, a contradiction.