Math 4853 homework

62. Prove that every uncountable subset of \( \mathbb{R} \) has a limit point in \( \mathbb{R} \). (Let \( A \) be an uncountable subset of \( \mathbb{R} \), and for \( n \in \mathbb{Z} \) put \( A_n = A \cap [n, n+1] \).)

63. Let \( \{ x_n \} \) be a sequence in a metric space \( X \). Prove that if \( x_n \to x \), then \( \{ x_n \} \) is Cauchy.

64. Give \( \mathbb{R}^k \) the metric \( d(x, y) = \| x - y \| \). Let \( \{ z_n \} \) be a sequence of points in \( \mathbb{R}^k \), written in coordinates as \( z_n = (z^1_n, z^2_n, \ldots, z^k_n) \). Prove that \( \{ z_n \} \) is Cauchy if and only if each \( \{ z^i_n \} \) is a Cauchy sequence in \( (\mathbb{R}, |x - y|) \).

65. Let \( \{ f_n \} \) be a sequence of functions in \( C([0, 1], \mathbb{R}^k) \) (the set of continuous functions from \([0, 1]\) to \( \mathbb{R}^k \)). Prove that if \( \{ f_n \} \to f \) uniformly, then \( \{ f_n \} \to f \) pointwise.

66. Let \( f_n : [0, 1] \to \mathbb{R} \) be \( f_n(x) = x^n \), and let \( f : [0, 1] \to \mathbb{R} \) be defined by \( f(x) = 0 \) if \( x < 1 \) and \( f(1) = 1 \). Using the definitions, prove that \( f_n \to f \) pointwise but not uniformly.