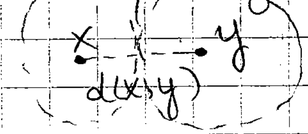
$\boxed{\text{Ex}}$ If X is an infinite set, then $(X, \text{cofinite})$ is not metrizable

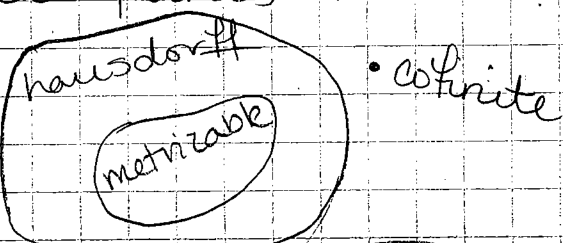
Pf: Lemma: metrizable spaces are Hausdorff
- let Y be metrizable, so Y has the metric topology for some metric d
- let $x, y \in Y$, $x \neq y$



Claim $B(x, \frac{d(x,y)}{2})$ and $B(y, \frac{d(x,y)}{2})$ are disjoint neighborhoods of x & y

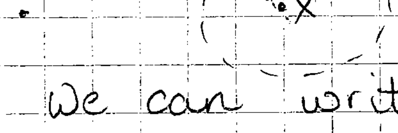
Suppose for contradiction that some z is in both balls. then $d(x,y) \leq d(x,z) + d(z,y)$
$$< \frac{d(x,y)}{2} + \frac{d(x,y)}{2} = d(x,y)$$
which is a contradiction. ■

But if X is infinite, then any two nonempty open sets intersect:
Suppose $U = X - F_U$ and $V = X - F_V$ are open, so F_U and F_V are finite
$U \cap V = X - (F_U \cup F_V) \neq \emptyset$
So no two points have disjoint neighborhoods



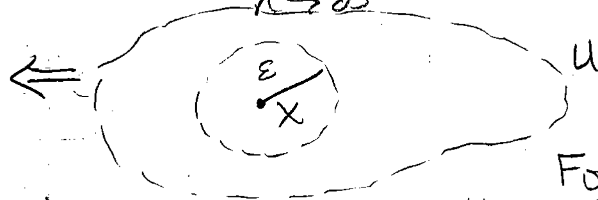
Sequences and limit points

def: let X be a topological space
let $\{x_n\}_{n=1}^{\infty}$ a sequence of points in X
(i.e. a function $f: \mathbb{N} \to X$, x_n denotes $f(n)$)
We say a sequence $\{x_n\}$ converges to $x \in X$ if $\forall$ nbd U of x, $\exists N$, $n \geq N \Rightarrow x_n \in U$



We can write $\lim x_n = x$ or $\{x_n\} \to x$

In $\mathbb{R}^n$, a sequence converges to $x \Leftrightarrow \lim_{n\to\infty} x_n = x$ in the ε-δ sense

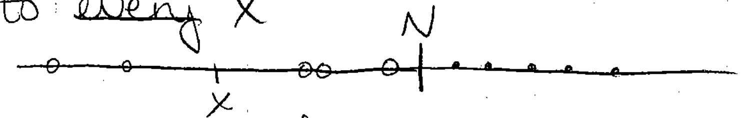
$\Leftarrow$ 

$\exists \varepsilon$ s.t. $B(x,\varepsilon) \subseteq U$
$\exists N$ s.t. $n \geq N$, $\|x_n - x\| < \varepsilon$
For $n \geq N$, $x_n \in B(x,\varepsilon) \subseteq U$

$\Rightarrow$ given ε, choose $U = B(x,\varepsilon)$

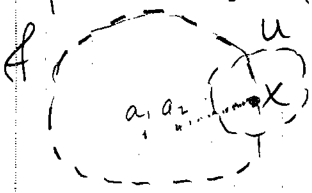
- In Hausdorff spaces, limits of sequences are unique: if $\{x_n\} \to x$ and $\{x_n\} \to y$, then $x = y$

- $\boxed{Ex}$ In ($\mathbb{R}$, cofinite), the sequence $\{n\} = \{1, 2, \ldots\}$ converges to <u>every</u> x



<u>Pf</u>: fix x.
let U be a nbd of x. $U = \mathbb{R} - \{x_1, \ldots, x_k\}$
Choose $N > \max\{x_1, \ldots, x_k\}$
If $n > N$, then $n \neq x_i$, so $n \in U$

<u>Proposition</u> let $A \subseteq X$. let $\{a_n\}$ be a set of points in A. if $\{a_n\} \to x$, then $x \in \overline{A}$



<u>Pf</u>: let U be any nbd of x
It must contain some a_n's (in fact, all a_n's for n greater than some n). So every nbd of x contains a point of A. $\therefore x \in \overline{A}$ ∎

OR contradiction: Suppose $x \notin \overline{A}$.
The $X - \overline{A} = U$ is a nbd of x containing no point of A

April 21 $\{x_n\} \to x$ means $\forall$ nbd U of x,
$\exists N$, $n > N \Rightarrow x_n \in U$

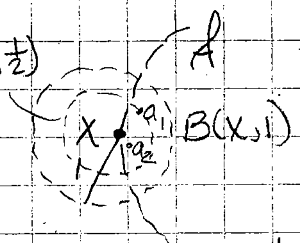
"Every neighborhood of x contains all but finitely many x_n"

- In Hausdorff spaces limits are unique
- Metrizable spaces are hausdorff

Sequence lemma:
let X be a metrizable space and $A \subseteq X$.
If $x \in \bar{A}$, then there exists a sequence $\{a_n\}$ of points in A with $\{a_n\} \to x$

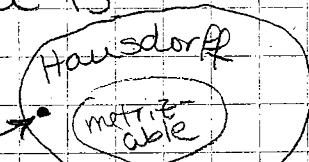
A x

pf: let d be a metric that induces the topology on X.
For each $n \in \mathbb{N}$, choose
$B(x, \frac{1}{n})$ is a nbd of x,
so it contains some point $a_n \in A$.



We claim that $\{a_n\} \to x$. let U be a nbd of x. There exists, by the "handy lemma", there exists $\varepsilon > 0$ with $B(x,\varepsilon) \subseteq U$.
for every $n > \frac{1}{\varepsilon}$, $\frac{1}{n} < \varepsilon$, so $a_n \in B(x, \frac{1}{n}) \subseteq B(x,\varepsilon) \subseteq U$

$\hookrightarrow$ gives an example of a space that is Hausdorff, but not metrizable

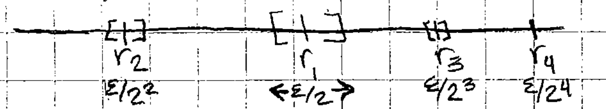


Ex a nonmetrizable Hausdorff space:

$\cdot q$

let $X = \mathbb{R} \cup \{q\}$ ———— $\mathbb{R}$ with the following topology: U is open if either $q \notin U$ (so $\mathbb{R}$ has discrete top.) or $X - U$ is countable (including finite $X - U$)

Note: Countable subsets of $\mathbb{R}$ are "small":
Suppos $\{r_1, r_2, \ldots\}$ is a countable subset of $\mathbb{R}$



let $\varepsilon > 0$
$\bigcup_{n=1}^{\infty}$ of these intervals $= S$
$\{r_1, r_2, \ldots\} \subseteq S$
total length of $S \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$

back to Ex... $X = \mathbb{R} \cup \{q\}$

U is open if either $q \notin U$ or $X - U$ is countable

① X is Hausdorff

Pf: let $x, y \in X$ with $x \neq y$

Case 1: one of x or y is q, say $x = q$

$\{y\}$ is an open set and $X - \{y\}$ is open.

So they are disjoint neighborhoods of x & y

case 2: $x, y \in \mathbb{R}$, then $\{x\}$ and $\{y\}$ are disjoint nbd

② $q \in \bar{A}$. let $A = \mathbb{R}$

let U be any neighborhood of q. $U \neq \{q\}$

Since $X - U$ is countable, so U contains a point of $\mathbb{R}$.

③ there is no sequence of points in A that converges to q. (fails sequence lemma)

Pf: Suppose $\{a_n\}$ is any sequence of points in A. let $U = \{q\} \cup (\mathbb{R} - \{r \in \mathbb{R} \mid r = a_n \text{ for some } n\})$

So $X - U = \{r \in \mathbb{R} \mid r = a_n \text{ for some } n\}$,

so U is open in X.

So U is a nbd of q containing no a_n.

$\therefore \{a_n\}$ does not converge to q.

* By the Sequence Lemma, X is not metrizable

Thm.: let X be a compact space. If A is an infinite subset of X, then A has a limit point.

($\mathbb{N}$ is an infinite subset of $\mathbb{R}$ w/ no lim. pt., so this is false in noncompact spaces. Similarly, $\{\frac{1}{n}\}$ in $(0,1]$)

Pf: We will show that $A' = \emptyset \Rightarrow A$ is finite.

Assume A is infinite & $A' = \emptyset$. $\bar{A} = A \cup A' = A \cup \emptyset = A$, so A is a closed subset of X.

For each $a \in A$, $\exists$ a nbd U_a s.t. $U_a \cap A = a$

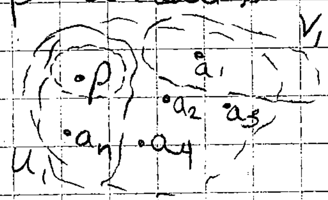
$X - A$ is open, so $\{X - A\} \cup \{U_a\}_{a \in A}$ is an open cover of X. ($A \subseteq \bigcup_{a \in A} U_a$). Since X is compact, there is a finite subcover. $X = \underbrace{X - A}_{\text{contains no pt. of } A} \cup \underbrace{U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n}}_{\text{contain 1 pt. ea. of } A \text{ plus pt.s not in } A} \supseteq A$

So $A = \{a_1, \ldots, a_n\}$

$\therefore A$ is finite

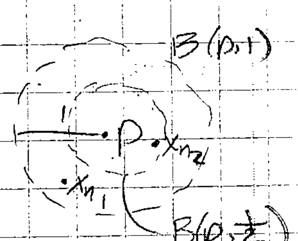
Thm. Let X be a compact metrizable space (for ex., any compact subset of a metric space). Then every sequence in X has a convergent subsequence.

Lemma: Let Y be a Hausdorff space, $A \subseteq Y$, and $p \in A'$. Then every nbd of p contains infinitely many points of A.



Pf: Suppose some nbd, U, of p has $U \cap A$ finite, $(U \cap A) - \{p\} = \{a_1, \ldots, a_n\}$
For each $1 \leq i \leq n$ choose disjoint nbds U_i of p and V_i of a_i
Then $U_1 \cap U_2 \cap \cdots \cap U_n$ is a nbd of p that is disjoint from each V_i
(since $U_1 \cap \cdots \cap U_n \subseteq U_i$) so it contains no a_i
So $(U \cap U_1 \cap \cdots \cap U_n) \cap A \subseteq \{p\}$ (depending on whether p is in A).
$\therefore p \notin A'$

Pf of Thm: Let $\{x_n\}$ be a sequence in X.
Let $A = \{x \mid x = x_n$ for some $n\}$
- If A is finite, then x_n has a constant subsequence, which is convergent (inf. terms, but finitely many pts.)
- Suppose A is infinite. Then $\exists p \in A'$.



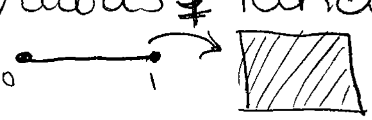
In $B(p,1)$, choose some x_{n_1}
$B(p,\frac{1}{2})$ contains infinitely many points of A (by the lemma).
So there is some x_{n_2} with $n_2 > n_1$ and $x_{n_2} \in B(p, \frac{1}{2})$.
So inductively, choose $x_{n_k} \in B(p, \frac{1}{k})$ w/ $n_k > n_{k-1}$
Then $\{x_{n_k}\} \to p$.

Corollary: Every bounded sequence of $\mathbb{R}$ has a convergent subsequence

Pf: Every bounded sequence lies in some closed interval $[-M, M]$, which is compact.

– End of material for test 3 –

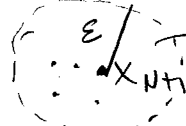
[Goal: There exists a continuous surjective function from $[0,1]$ to $[0,1]\times[0,1]$ (but no differentiable function)]



Complete Metric Spaces:

def: let (X,d) be a metric space. A sequence $\{x_n\}$ in X is cauchy if $\forall \varepsilon>0$, $\exists N$, if $m,n>N$, then $d(x_m,x_n)<\varepsilon$.

x_{N+1}, x_{N+2}



all other sequence terms lie in here (etc.)

• (X,d) is complete when every cauchy sequence converges

usual distance

$\boxed{Ex}$ 1: $(\mathbb{R}, |x-y|)$ is complete

pf: let $\{x_n\}$ be a cauchy sequence

Step 1: Show $\{x_n\}$ is bounded

Choose N so that if $m,n>N$, then $|x_m-x_n|<1$

let $M=\max\{|x_1|,|x_2|,\ldots,|x_{N+1}|\}+1$

If $n\le N$, then $|x_n|<M$

If $n>N$, then $|x_n|-|x_{N+1}|\le|x_n-x_{N+1}|<1$

$|x_n|<|x_{N+1}|+1\le M$

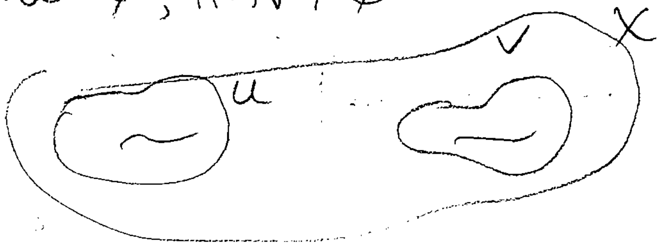
So every $x_n\in[-M,M]$

$A\subseteq X$

$\exists$ disjoint open subsets U & V of X

$A\subseteq U\cup V$

$A\cap U\neq\emptyset$, $A\cap V\neq\emptyset$



[$X=U\cup V$ disjoint open sets

if A is a connected subset of X, then $A\subseteq U$ or $A\subseteq V$ (otherwise, $A=(A\cap U)\cup(A\cap V)$ disjoint, nonempty open in A)]

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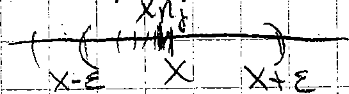
(X,d) is complete if every Cauchy sequence in (X,d) converges

Ex[1] $(\mathbb{R}, |x-y|)$ is complete
(there is a number where the seq. piles up)

<u>Pf</u>: let $\{x_n\}$ be a Cauchy sequence
- Step 1: x_n is bounded
- Step 2: Since $\{x_n\}$ is bounded, it lies in some closed interval $[M, M]$.

* <u>This is compact</u>, so $\{x_n\}$ has a convergent subsequence. Say $\{x_{n_i}\} \to x$
- Step 3: Show the entire sequence $\{x_n\} \to x$

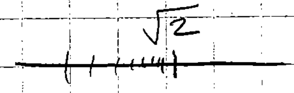
let $\varepsilon > 0$, $\exists N_1$, if $n_i > N_1$, then $|x_{n_i} - x| < \varepsilon/2$
$\exists N_2$, if $m,n > N_2$, $|x_m - x_n| < \varepsilon/2$



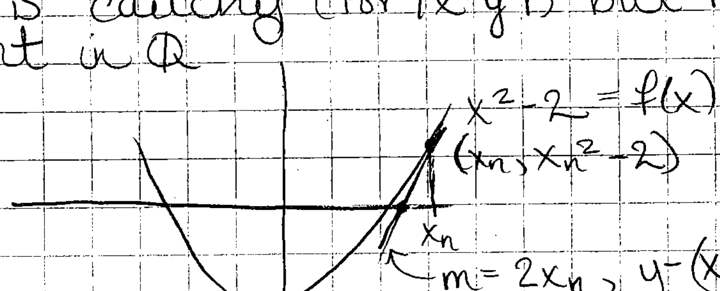
Choose $n_j > \max\{N_1, N_2\}$,
$$|x_n - x| \le |x_n - x_{n_j}| + |x_{n_j} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Ex[2] $(\mathbb{Q}, |x-y|)$ is NOT complete

<u>Pf</u>: let x_1 be a positive rational number
define $x_2 = \frac{x_1}{2} + \frac{1}{x_1} \in \mathbb{Q}$



inductively, define $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$
$\{x_n\}$ is a Cauchy sequence in $\mathbb{Q}$ that does not converge (to any element of $\mathbb{Q}$)
<u>reason</u>: in $\mathbb{R}$, $\{x_n\} \to \sqrt{2}$
so $\{x_n\}$ is cauchy (for $|x-y|$) but not convergent in $\mathbb{Q}$



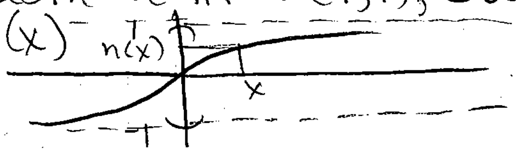
$m = 2x_n$, $y - (x_n^2 - 2) = m(x - x_n)$
$y = 2x_n x - x_n^2 - 2$

<u>(Newton's Method)</u>

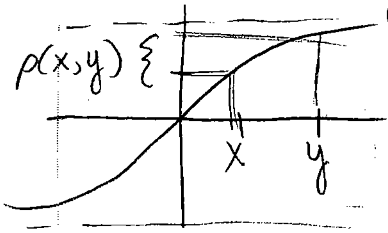
when $0 = 2x_n x - x_n^2 - 2$,
$$x = \frac{x_n}{2x_n} + \frac{2}{2x_n}$$
$$x = \frac{x_n}{2} + \frac{1}{x_n}$$

$\boxed{Ex}^3$ An incomplete metric on $\mathbb{R}$ whose metric topology is the standard topology (so completeness is a "metric property", not a "topological property")

Choose a homeomorphism $h: \mathbb{R} \to (-1,1)$, such as
$h(x) = \frac{2}{\pi}\arctan(x)$
$h^{-1}(x) = \tan(\frac{\pi}{2}x)$



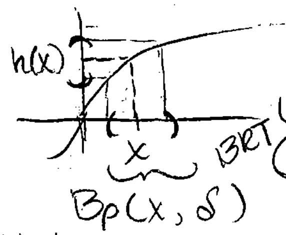
define a metric ρ on $\mathbb{R}$ by $\rho(x,y) = |h(x)-h(y)|$



check metric prop.:
1.) $\rho(x,y) \geq 0$, $\rho(x,y) = 0 \Leftrightarrow h(x) = h(y)$
$\Leftrightarrow x = y$
2.) $\rho(x,y) = \rho(y,x)$
3.) $\rho(x,y) = |h(x)-h(y)| \leq |h(x)-h(z)| + |h(z)-h(y)|$
$= \rho(x,z) + \rho(z,y)$

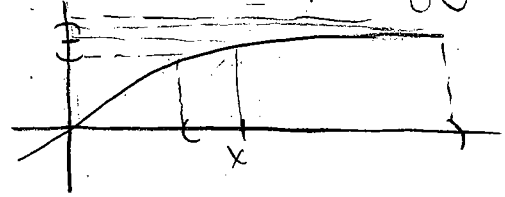
- ρ induces the standard topology on $\mathbb{R}$:

$\underline{Pf}$: 1.) $B_\rho(x,\delta) = \{z \mid |h(z)-h(x)| < \delta\}$
$= \{z \mid h(z) \in (h(x)-\delta, h(x)+\delta)\}$
$= h^{-1}((h(x)-\delta, h(x)+\delta))$

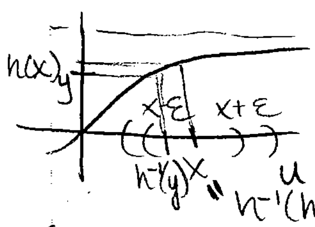


which is open b/c h is cont.
BRT ① (open in the standard topology)

let U be open in $\mathbb{R}$, and $x \in U$
(notice the balls become very off center)



$\exists \varepsilon$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq U$



Since h^{-1} is cont., $\exists \delta > 0$ s.t.
if $|y - h(x)| < \delta$, then $|h^{-1}(y) - h^{-1}(h(x))| < \varepsilon$
$|h^{-1}(y) - x| < \varepsilon$

Show $B_\rho(x,\delta) \subseteq U$

$x = h^{-1}(h(x))$ let $z \in B_\rho(x,\delta)$, $\rho(x,z) = |h(z) - h(x)| < \delta$

BRT ② $|z-x| = |h^{-1}(h(z)) - x| < \varepsilon$
$\therefore z \in (x-\varepsilon, x+\varepsilon) \subseteq U$

So by the $\underline{\text{Basis Rec. Thm.}}$, $\{B_\rho(x,\delta)\}$ is a basis for the standard topology & the ρ-metric topology

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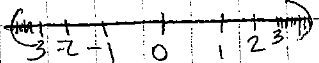
(3) An incomplete metric on $\mathbb{R}$ (inducing the standard topology)

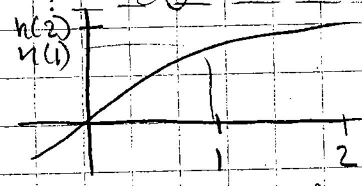


$\rho(x,y) \stackrel{\triangle}{=} |h(x)-h(y)|$

✓ 1. ρ is a metric

✓ 2. ρ induces the standard topology

3. $(\mathbb{R}, \rho)$ is not complete



Consider $\{n\}$ $(x_n = n)$

Check that it is Cauchy:

$\{h(n)\} \to 1$ in $\mathbb{R}$

$\therefore$ $h(n)$ is Cauchy (every conv. seq. in $\mathbb{R}$ is Cauchy)

let $\varepsilon > 0$, since $h(n)$ is Cauchy (for the usual metric),

$\exists N$, if $m, n \geq N$, then $|h(m) - h(n)| < \varepsilon$

$\rho(m,n) < \varepsilon$

$\therefore$ n is ρ-cauchy

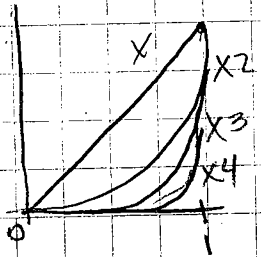
But $\{n\}$ does not converge

(convergence depends only on the topology, not the metric – the definition of convergence refers only to open sets)

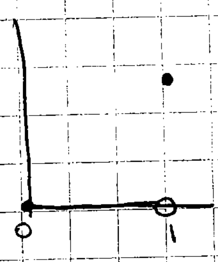
Summary: Completeness depends on the metric

$(\mathbb{R}, |\ -\ |)$ is complete

$(\mathbb{R}, \rho)$ is not complete



$\{x^n\} \to$

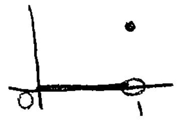


the limit is not continuous

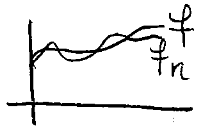
-control by making different senses of convergence cont. functions from $[0,1] \to \mathbb{R}$

④ <u>Uniform Convergence</u>:
let $\{f_n\}$ be a sequence in $\mathcal{C}([0,1])$

- We say $\{f_n\}$ converges <u>pointwise</u> when for each $x \in [0,1]$, $\{f_n(x)\} \to f(x)$

$\{x_n\} \to$  pointwise

- We say $\{f_n\} \to f$ uniformly if $\forall \varepsilon > 0, \exists N$, $n \geq N \Rightarrow \forall x \in [0,1]$, $|f_n(x) - f(x)| < \varepsilon$

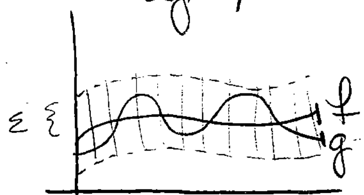


pointwise convergence is $\forall x \in [0,1], \forall \varepsilon > 0, \exists N$, $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$

If $\{f_n\} \to f$ uniformly, then it converges pointwise but not conversely

⑤ let ρ be the metric on $\mathcal{C}([0,1], \mathbb{R}^k)$ defined by
$\rho(f,g) = \max\limits_{x \in [0,1]} \{\|f(x) - g(x)\|\}$

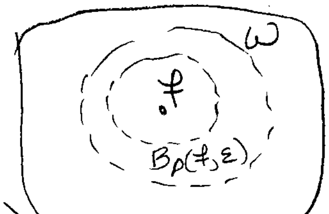
<u>Proposition</u>: $\{f_n\} \to f$ in the topology induced by $\rho \Leftrightarrow \{f_n\}$ converges to f <u>uniformly</u>



- $\rho(f,g) < \varepsilon$ means g lies in this band
- $B_\rho(f,\varepsilon) = \{g \mid \text{graph of } g \text{ lies in this band}\}$

$\Rightarrow$ <u>Pf</u>: Suppose $\{f_n\} \to f$ in the ρ-topology
let $\varepsilon > 0$, $\exists N$ s.t. $n \geq N$, then $f_n \in \underbrace{B_\rho(f,\varepsilon)}_{\text{a nbd of } f}$
so $\rho(f_n, f) < \varepsilon$
$\rho(f_n, f) = \max\limits_{x \in [0,1]} \|f_n(x) - f(x)\| < \varepsilon$

$\forall x \in [0,1]$, $\|f_n(x) - f(x)\| \leq \rho(f_n, f) < \varepsilon$

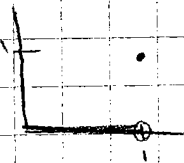


$\Leftarrow$ Assume $\{f_n\} \to f$ uniformly.
let W be any nbd of f
$\exists \varepsilon > 0$, $B_\rho(f,\varepsilon) \subseteq W$ (by Handy lemma)
$\exists N, n \geq N \Rightarrow \forall x \in [0,1]$, $\|f_n(x) - f(x)\| < \varepsilon$ (uniform conv.)
$\Rightarrow \max\limits_{x \in [0,1]} \|f_n(x) - f(x)\| < \varepsilon$
$\Rightarrow \rho(f_n, f) < \varepsilon \Rightarrow f_n \in B_\rho(f,\varepsilon) \subseteq W$

fix an integer K

$f_n \in \mathcal{C}([0,1], \mathbb{R}^K)$

$\{f_n\} \to f$ pointwise means $\forall x \in [0,1]$,
$\{f_n(x)\} \to f(x)$

Ex $\underbrace{\{x^n\}}_{\text{in } \mathcal{C}([0,1], \mathbb{R})} \longrightarrow$  not in $\mathcal{C}([0,1], \mathbb{R})$

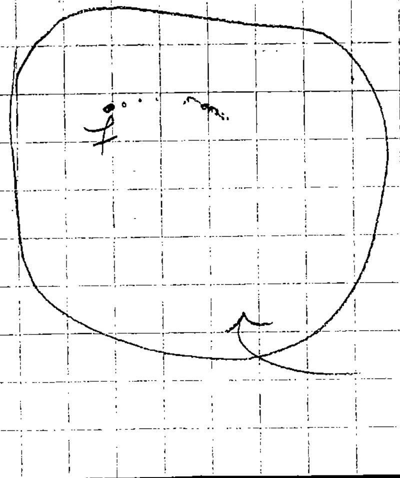
$\{f_n\} \to f$ uniformly means $\forall \varepsilon > 0$, $\exists N$, $n \geq N$,
$\|f_n(x) - f(x)\| < \varepsilon$

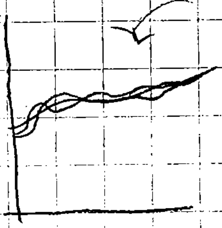
Thm: Uniform Covergence Thm.

Suppose $\{f_n\}$ is a sequence of functions in $\mathcal{C}([0,1], \mathbb{R}^K)$. If $\{f_n\} \to f$ uniformly, then f is continuous

Pf: Let $x_0 \in [0,1]$. Let $\varepsilon > 0$. $\exists N$ so that if $n \geq N$, $\|f_n(x) - f(x)\| < \varepsilon/3$ for all $x \in [0,1]$
Choose $n_0 > N$. f_{n_0} is continuous, so $\exists \delta > 0$, if $x - x_0 < \delta$, then $\|f_{n_0}(x) - f_{n_0}(x_0)\| < \frac{\varepsilon}{3}$
For $|x - x_0| < \delta$, $\|f(x) - f(x_0)\| \leq \|f(x) - f_{n_0}(x)\| + \|f_{n_0}(x) - f_{n_0}(x_0)\| + f_{n_0}(x_0) - f(x_0)\|$
$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ ∎

Thm: For $f, g \in \mathcal{C}([0,1], \mathbb{R}^K)$ let $\rho(f,g) = \max_{x \in [0,1]} \|f(x) - g(x)\|$
(the uniform metric)
$(\mathcal{C}[0,1], \mathbb{R}^K, \rho)$ is complete





they are limiting to a cont. function

Take an $\mathbb{R}^K$ (standard metric)

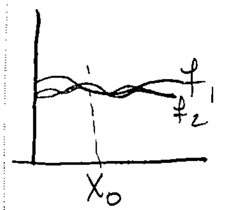
Notation: $[0,1]$ is denoted by I

$\mathcal{C}(I,\mathbb{R}^K)$ = the continuous functions from I to $\mathbb{R}^K$. $\rho(f,g) = \max_{x\in[0,1]} \|f(x)-g(x)\|$

<u>Thm.</u>: $(\mathcal{C}(I,\mathbb{R}^K),\rho)$ is complete

<u>pf</u>: let $\{f_n\}$ be a cauchy sequence in $\mathcal{C}(I,\mathbb{R}^K)$

<u>step 1</u>: Show for each point $x_0 \in I$, $\{f_n(x_0)\}$ is Cauchy in $\mathbb{R}^K$



Given $\varepsilon>0$, $\exists N$ s.t. $m,n>N$, then $\rho(f_m,f_n)<\varepsilon$

For $m,n>N$, $\|f_m(x_0)-f_n(x_0)\| \le \max_{x\in[0,1]} \|f_m(x)-f_n(x)\| = \rho(f_m,f_n) < \varepsilon$

<u>step 2</u>: $\mathbb{R}^K$ is complete

let $\{z_n\}$ be a Cauchy sequence in $\mathbb{R}^K$

Let $\pi_i : \mathbb{R}^K \to \mathbb{R}$ be the coordinate function

For $z,w \in \mathbb{R}^K$, $|\pi_i(z)-\pi_i(w)| \le \|z-w\|$

check: $\|z-w\| = \left(\Sigma(\pi_j(z)-\pi_j(w))^2\right)^{1/2} \ge \left((\pi_i(z)-\pi_i(w))^2\right)^{1/2} = |\pi_i(z)-\pi_i(w)|$

So for each i, $\{\pi_i(z_n)\}$ is Cauchy in $\mathbb{R}$

Since $\mathbb{R}$ is complete, $\{\pi_i(z_n)\} \to t_i$ for some t_i

$\therefore \{z_n\} \to (t_1, t_2, \dots, t_n)$

So by steps 1 & 2, each sequence $\{f_n(x_0)\}$ is convergent for $x_0 \in I$.

Define $f: I \to \mathbb{R}^K$ by $\{f_n(x)\} \to f(x)$. Note that we do not yet know that f is continuous

<u>step 3</u>: Show $\forall x_0 \in I$, if $m,n \ge N \Rightarrow \|f_m(x_0)-f_n(x_0)\| < \varepsilon$, then $n \ge N \Rightarrow \|f_n(x_0) - f(x)\| \le \varepsilon$

⋮

<u>Lemma</u>: Let $\{x_n\}$ be a convergent (hence Cauchy) sequence in a metric space, $\{x_n\} \to x$

If $m,n \ge N \Rightarrow d(x_m,x_n) < \varepsilon$, then $\forall n \ge N \Rightarrow d(x_n,x) \le \varepsilon$

<u>pf</u>: Suppose not, so $\exists n \ge N$ with $d(x_n,x) > \varepsilon_0 > \varepsilon$. for some ε_0

For all $m \ge N$, $\varepsilon_0 < d(x_n,x) \le d(x_n,x_m) + d(x_m,x)$

$d(x_m,x) \ge \varepsilon_0 - d(x_n,x_m) > \varepsilon_0 - \varepsilon > 0$

So $B(x,\varepsilon_0-\varepsilon)$ contains no x_m with $m \ge N$

This contradicts the fact that $\{x_n\} \to x$

...**Step 3**: By the Lemma, if $m,n \geq N \Rightarrow \|f_m(x_0), f_n(x_0)\| < \varepsilon$, then $n \geq N \Rightarrow \|f_n(x_0) - f(x_0)\| \leq \varepsilon$

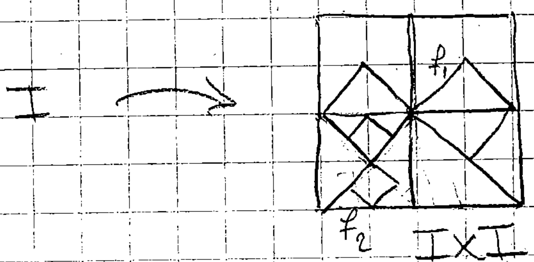
Step 4: Show $\{f_n\} \to f$ uniformly

pf: let $\varepsilon > 0$. $\exists N$, if $m,n \geq N$, then $\|f_m(x) - f_n(x)\| \leq \rho(f_m, f_n) < \frac{\varepsilon}{2}$ $\forall x \in I$

For each $x \in I$, if $m,n \geq N$, then $\|f_m(x) - f_n(x)\| \leq \rho(f_m, f_n) < \frac{\varepsilon}{2}$

so if $n \geq N$, $\|f_n(x) - f(x)\| \leq \frac{\varepsilon}{2} < \varepsilon$.

Step 5: By the uniform convergence Thm., f is continuous. So $f \in \mathcal{C}(I, \mathbb{R}^K)$



$f_n \in \mathcal{C}(I, \mathbb{R}^2)$

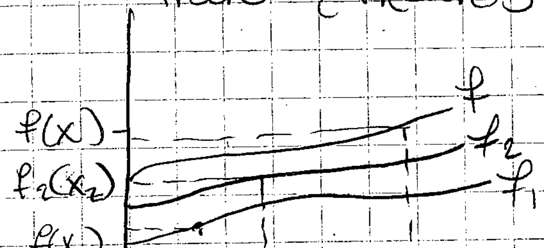
- we will prove $\{f_n\}$ is Cauchy, so by the thm., $f_n \to f$ for some $f \in \mathcal{C}(I, \mathbb{R}^2)$
- will prove $f(I) = I^2$

Lemma Let $\{f_n\}$ be a sequence in $\mathcal{C}(\underset{[0,1]}{I}, \mathbb{R}^K)$

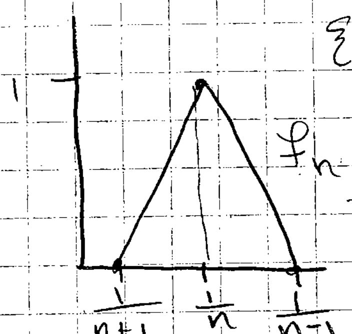
Suppose 1.) $\{f_n\} \to f$ uniformly, and

2.) $\{x_n\} \to x$ in I

Then $\{f_n(x_n)\} \to f(x)$



- this can fail if $\{f_n\}$ converges to f only pointwise, even when f is continuous.



$\{\frac{1}{n}\} \to 0$

$\{f_n(\frac{1}{n})\} = \{1, 1, \ldots\} \to 1$

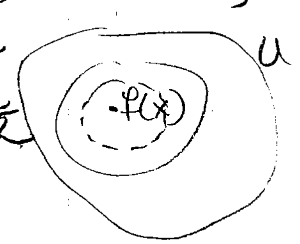
$\{f_n\} \to Z$ the zero function pointwise $Z(0) = 0 \neq 1$

pf of Lemma:

Let U be a nbd of $f(x)$ in $\mathbb{R}^k$ and choose $\varepsilon > 0$ with $B(f(x), \varepsilon) \subseteq U$

By the uniform convergence Thm, f is cont.,
$\exists \delta > 0$, if $|y-x| < \delta$, then $\|f(y) - f(x)\| < \frac{\varepsilon}{2}$

Since $\{f_n\} \to f$ uniformly,



$\exists N_1$, $n > N_1$, for all $y \in I$, $\|f_n(y) - f(y)\| < \frac{\varepsilon}{2}$
$\exists N_2$, $n > N_2$, $|x_n - x| < \delta$

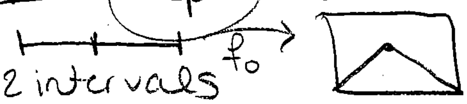
For all $n > \max\{N_1, N_2\}$

1. $|x_n - x| < \delta$, so $\|f(x_n) - f(x)\| < \frac{\varepsilon}{2}$
2. $\|f_n(x_n) - f(x_n)\| < \frac{\varepsilon}{2}$
3. $\|f_n(x_n) - f(x)\|$
$\le \|f_n(x_n) - f(x_n)\| + \|f(x_n) - f(x)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

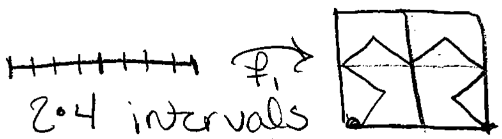
So $f_n(x_n) \in B(f(x), \varepsilon) \subseteq U$ ■

Thm: There exists a continuous surjection $f: I \to I \times I$

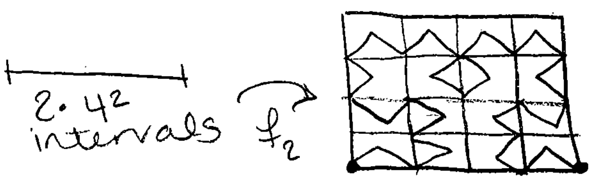
pf: Step 1: construct a sequence $\{f_n\}$ in $\mathcal{C}(I, \mathbb{R}^2)$



- linear on each interval

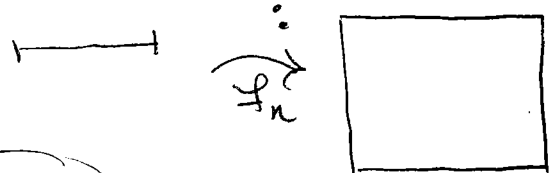


- 4 squares of side $\frac{1}{2}$



- 4^2 squares of side $\frac{1}{2^2}$

⋮



- 4^n squares of side $\frac{1}{2^n}$

Step 2: Prove that $\{f_n\}$ is cauchy (for ρ)

Observe that for each x and each n, $f_n(x)$ and $f_{n+1}(x)$ lie in the same one of the 4^n squares of side $1/2^n$

So $\|f_n(x) - f_{n+1}(x)\| \le \sqrt{2}\left(\frac{1}{2^n}\right)$

For $m > n$ and all $x \in I$, $\|f_m(x) - f_n(x)\|$
$\le \|f_m(x) - f_{m-1}(x)\| + \|f_{m-1}(x) - f_{m-2}(x)\| + \dots + \|f_{n+1}(x) - f_n(x)\|$
$\le \sqrt{2}\left(\frac{1}{2^{m-1}}\right) + \sqrt{2}\left(\frac{1}{2^{m-2}}\right) + \dots + \sqrt{2}\left(\frac{1}{2^n}\right) < \sqrt{2} \sum_{k=n}^{\infty} \frac{1}{2^k}$

$$\dots < \sqrt{2} \sum_{k=n}^{\infty} \frac{1}{2^k} = \sqrt{2}\, \frac{1}{2^{n-1}}$$

Given $\varepsilon > 0$, choose N so that $\frac{\sqrt{2}}{2^{N-1}} < \varepsilon$

For all $m,n > N$, say $m > n$,

We have $\rho(f_m, f_n) = \max_{x \in I} \|f_m(x) - f_n(x)\| \le \frac{\sqrt{2}}{2^{n-1}} < \frac{\sqrt{2}}{2^{N-1}} < \varepsilon$.

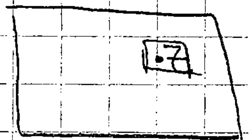
Step 3: Since $\mathcal{C}(I, \mathbb{R}^k)$ is complete, $\{f_n\} \to f$ uniformly with $f \in \mathcal{C}(I, \mathbb{R}^2)$

For each x, $\{f_n(x)\} \to f(x)$

So $f(x)$ is a limit of points in $I \times I$,

Since $I \times I$ is closed, $f(x) \in I \times I$

So we have $f : I \to I \times I$ is continuous



Step 4: Prove f is surjective:

let $z \in I \times I$

for each n, z lies in at least one of the 4^n squares of side $\frac{1}{2^n}$, so

$\exists\, x_n \in I$ with $f_n(x_n)$ in that same square

Since $\|f_n(x_n) - z\| \le \sqrt{2}\left(\frac{1}{2^n}\right)$, $\{f_n(x_n)\} \to z$

Since I is compact and metrizable, $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\} \to x$. Still, $\{f_{n_j}(x_{n_j})\} \to z$

By the lemma, $\{f_{n_j}(x_{n_j})\} \to f(x)$

Since $I \times I$ is Hausdorff, limits are unique,

So $f(x) = z$. ■