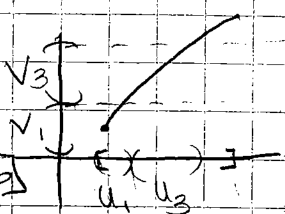
April 5

Thm: Every cont. function from $[a,b]$ to $\mathbb{R}$ is bounded.



Pf: let $V_n = (n-1, n+1)$, let $f^{-1}(V_n) = U_n$

$$\bigcup_{n\in\mathbb{Z}} U_n = \bigcup_{n\in\mathbb{Z}} f^{-1}(V_n) = f^{-1}\Big(\bigcup_{n\in\mathbb{Z}} V_n\Big) = f^{-1}(\mathbb{R}) = [a,b]$$

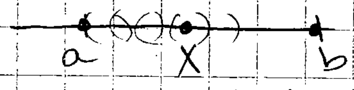
and U_n is open because f is continuous

Summary

1. every $x \in [a,b]$ lies in some U_n, so $\{U_n\}_{n\in\mathbb{Z}}$ is a collection of open subsets of $[a,b]$ whose union is $[a,b]$ (an "open cover" of $[a,b]$)
2. We will show $\exists$ a finite subcollection $\{U_1, U_2, \ldots, U_{n_k}\}$ whose union is X (a "finite subcover"). Then, $[a,b] = \bigcup_{i=1}^{k} U_{n_i} = \bigcup_{i=1}^{k} f^{-1}(V_{n_i}) = f^{-1}\big(\bigcup_{i=1}^{k} (V_{n_i})\big)$

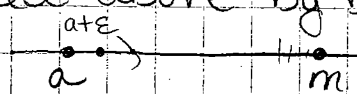
- So, $f([a,b]) \subseteq \bigcup_{i=1}^{k} V_{n_i}$. Since there are only finitely many, this is a bounded subset of $\mathbb{R}$, So f is bounded.

3. let $S = \{x \in [a,b] \mid \exists$ a finite subcollection of $\{U_n\}$ whose union contains $[a,x]\}$



$a \in S$ since a is in some U_n $\;[a,a]$

S is bounded above by b. S has a least upper bound, m.



$m > a$ since if $a \in U_{n_0}$, Then some $[a, a+\varepsilon) \in U_{n_0}$

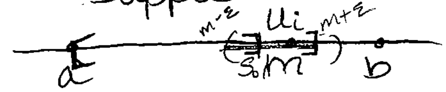
so $[a, a+\frac{\varepsilon}{2}] \subseteq U_{n_0}$

So $a + \frac{\varepsilon}{2} \in S$

So $a + \frac{\varepsilon}{2} \le m$

Step 1: Want to show $m=b$.

Suppose for contradiction that $m<b$



For some i, $m\in U_i$ so for some ε, $(m-\varepsilon, m+\varepsilon)\subseteq U_i$

$m-\varepsilon$ is not an upper bound for S, since m is the least upper bound.

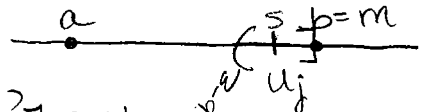
So $\exists s_0$ s.t. $m-\varepsilon<s_0\leq m$

$\therefore [a,s_0]\subseteq U_{n_1}\cup U_{n_2}\cup\cdots\cup U_{n_K}$ for some finite subcollection.

$[a, m+\frac{\varepsilon}{2}] = [a,s_0]\cup(m-\varepsilon, m+\frac{\varepsilon}{2}]\subseteq U_{n_1}\cup\cdots\cup U_{n_K}\cup U_i$

$\therefore m+\frac{\varepsilon}{2}$ is in S, which contradicts the fact that m is an upper bound of S ※

$\therefore m=b$



Step 2: $b\in U_j$ for some U_j,

so for some ε, $(b-\varepsilon, b]\subseteq U_j$

$b-\varepsilon$ is not an upper bound for S, so there is some $x\in S$, $b-\varepsilon<s\leq b$.

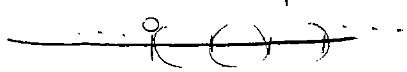
$[a,b] = [a,x]\cup(b-\varepsilon, b]\subseteq U_{n_1}\cup\cdots\cup U_{n_r}\cup U_j$ for some finite subcollection ∎

★ def: let X be a top. space, An open cover of X is a collection of open sets $\{U_\alpha\}_{\alpha\in a}$ in X s.t. $\bigcup_{\alpha\in a} U_\alpha = X$

★ a finite subcover of $\{U\}_{\alpha\in a}$ is a finite subcollection $\{U_{\alpha_1},\ldots U_{\alpha_n}\}$ s.t. $\bigcup_{i=1}^{n} U_{\alpha_i} = X$

★ X is compact if every open cover of X has a finite subcover.

$\boxed{Ex}^1$ $\mathbb{R}$ has finite open covers $\{(-\infty,1),(-1,\infty)\}$
The Union is $\mathbb{R}$, BUT $\mathbb{R}$ is not compact because $\{(n-1, n+1)\}_{n\in\mathbb{Z}}$



has no finite subcover.

$\therefore$ $\mathbb{R}$ is not compact

$\boxed{Ex}^2$ Every closed interval, $[a,b]$ in $\mathbb{R}$ is compact

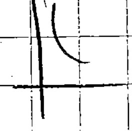
April 7

(3) If X is compact, then every continuous function $f: X \to \mathbb{R}$ is bounded
($\exists M$, $|f(x)| \le M$ for all $x \in X$)

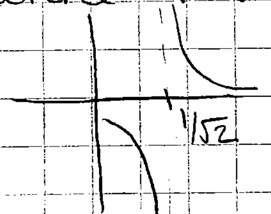
(4) $(0,1)$ is not compact
1.) let $U_n = (\frac{1}{n}, 1)$ $\quad (0,1) = \bigcup_{n=2}^{\infty} U_n$ is an open cover of $(0,1)$

BUT, for any finite subcollection
$\{U_{n_1}, \dots, U_{n_k}\}$, $\bigcup_{i=1}^{k} U_{n_i} = \left(\frac{1}{\max\{n_i\}}, 1\right) \neq (0,1)$

2.) $f(x) = \frac{1}{x}$ is continuous and unbounded on $(0,1)$

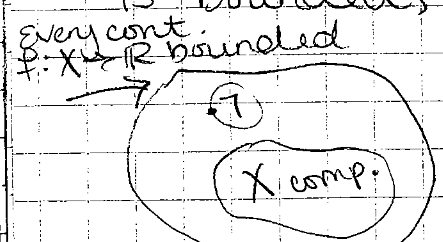


(5) $[0,1] \cap \mathbb{Q}$ is not compact $\quad 0 \quad 1/\sqrt{2}$
define $f(x) = \frac{1}{x - 1/\sqrt{2}}$ is cont. and unbounded

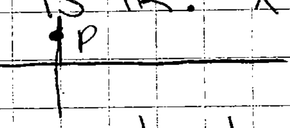


(6) In $(\mathbb{R}, \mathcal{L})$ the interval $[0,1]$ is not compact
1.) there is a cont. unbounded function $f: X \to \mathbb{R}$
2.) let $U_n = [0, 1 - \frac{1}{n}]$, $\quad 0 \quad 1-\frac{1}{n} \quad 1$
open subsets of X
$\{1\} = X \cap [1,2]$ so $\{1\}$ is open in X
$\{\{1\}\} \cup \{U_n \mid n \ge 2\}$ is an open cover with no finite subcover

(7) An example of an X s.t. every cont. $f: X \to \mathbb{R}$ is bounded, but X is not compact

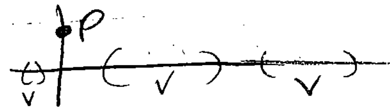


X is "adding a point whose closure is $\mathbb{R}$" $X = \mathbb{R} \cup \{p\}$, $p = (0,1)$



Note: does not have the subsp. top. as a subset of $\mathbb{R}^2$

U is open in $X \iff U$ is empty $(U = \emptyset)$ or $U = \{p\} \cup V$ for some open subset $V \subseteq \mathbb{R}$



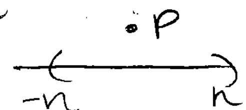
notice $\{p\}$ is open since $\{p\} = \{p\} \cup \emptyset$

① $\emptyset$ is open, X is open b/c $X = \{p\} \cup \mathbb{R}$

② $\bigcup \{p\} \cup V_\alpha = \{p\} \cup (\bigcup V_\alpha)$ (abbreviated argument)

③ $\bigcap \{p\} \cup V_{n_i} = \{p\} \cup (\bigcap V_{n_i})$

Claim: X is not compact



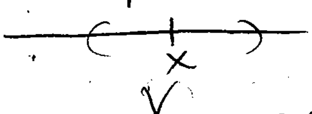
pf: let $U_n = \{p\} \cup (-n, n)$
$\{U_n\}$ is an open cover of X with no finite subcover

[Every continuous function $f: X \to \mathbb{R}$ is bounded (in fact, it is constant)!]

pf: Step 1: $\overline{\{p\}} = X$

pf: Certainly $\overline{\{p\}} \subseteq X$. Let $x \in X$ w/ $x \neq p$
let W be any neighborhood of x
then W is not empty,
so $W = \{p\} \cup V$ for some V open in $\mathbb{R}$

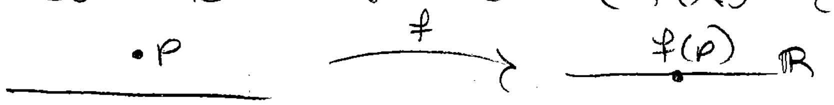
$S = \{p\}$



$p \in W$, so W contains a point of $\{p\}$
$\therefore x \in \{p\}' \subseteq \overline{\{p\}}$

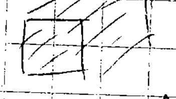
Step 2: let $f: X \to \mathbb{R}$ be continuous
$f(X) = f(\overline{\{p\}}) \subseteq \overline{f\{p\}}$ b/c f is continuous
$= \overline{f\{p\}} = \{f(p)\}$ ← points are closed subsets of $\mathbb{R}$

So f is constant $(f(X) \subseteq \{f(p)\})$



Aubrey

(8) A product of finitely many compact spaces is compact
eg. $[0,1]^n$, "the n-cube" is compact



- not difficult to prove

☆ (9) <u>Tychonoff Theorem</u>: any product of compact spaces is compact (even infinitely many)
<u>Pf</u> is very difficult, uses the Axiom of Choice in a very subtle way

[April 9:] Example revisited: $X = \underset{\mathbb{R}}{\overset{\cdot p}{\overline{\qquad\quad}}}$

X open in $X \iff U = \phi$ or $U = \{p\} \cup V$, V open in $\mathbb{R}$
$\overline{\{p\}} = X$ (where $\mathbb{R}$ is "stuck" to the point p)

- Another argument that every continuous $f: X \to \mathbb{R}$ is constant
$\{f(p)\}$ is a closed subset of $\mathbb{R}$
$f^{-1}(\{f(p)\})$ is a closed subset of X
$\{p\} \subseteq f^{-1}(\{f(p)\})$
$\therefore \overline{\{p\}} \subseteq f^{-1}(\{f(p)\})$
So $f(X) \subseteq \{f(p)\}$, i.e. f is constant

(10) Proposition: let $f: X \to Y$ be continuous
If X is compact, then its image is compact.
[$f(X)$ is a subset of Y $\{f(x) \mid x \in X\}$
$\boxed{X} \xrightarrow{f} \boxed{f(X) \; Y}$ it is a space when we give it the subspace top. as a subspace of Y.
So we often speak of "compact subsets" of a space]

In particular, if f is surjective, then Y is compact

<u>Pf</u>: let $f: X \to f(X)$ be continuous & it is surjective
let $\{U_\alpha\}_{\alpha \in a}$ be an open cover of $f(X)$.
$\{f^{-1}(U_\alpha)\}_{\alpha \in a}$ is an open cover of X.
Let $\{f^{-1}(U_{\alpha_i})\}_{i=1}^{n}$ be a finite subcover, so $X = \bigcup_{i=1}^{n} f^{-1} U_{\alpha_i}$
<u>Claim</u>: $f(X) = \bigcup_{i=1}^{n} U_{\alpha_i}$
$\supseteq$ OK
$\subseteq$ let $y \in f(x) \Rightarrow y = f(x)$ for some $x \in X$
$x \in f^{-1}(U_{\alpha_j})$ for some j, so $y = f(x) \in U_{\alpha_j}$. So $\{U_{\alpha_i}\}_{i=1}^{n}$ is fin. subcover

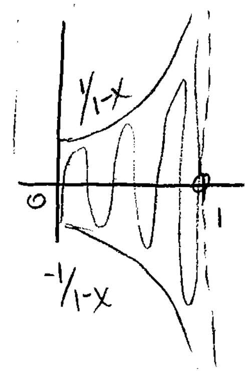
(took an open cover, and found a finite subcover)

Example of Use:

1.) $f: [0, 2\pi] \to S^1$ $f(t) = (\cos(t), \sin(t))$
is a continuous surjection

$[0, 2\pi]$ is compact, so S^1 is compact.

2.) There is no cont. surjection from $[0,1] \to [0,1)$
b/c $[0,1)$ would have to be compact.
Note: There is a cont. surjection from $[0,1) \to \mathbb{R}$

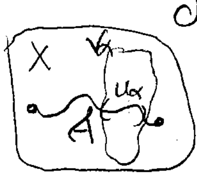


$f(x) = \dfrac{\sin(\frac{1}{1-x})}{1-x}$

(half open interval is much larger than compact sets)

(11) Thm.: Let X be compact. Let A be a closed subset of X. Then, ~~A is compact~~

Pf:



let $\{U_\alpha\}_{\alpha \in a}$ be an open cover of A. ~~for each~~ α, choose an open $V_\alpha \in X$ s.t. $V_\alpha \cap A = U_\alpha$
$X - A$ is an open subset of X, b/c A is closed.
$\{X-A\} \cup \{V_\alpha\}_{\alpha \in a}$ is an open cover of X
Now X is compact, so it has a finite subcover.
By adding the subset $X-A$, we may assume $X-A$ is one of the sets in this finite subcover
$\{X-A\} \cup \{V_{\alpha_i}\}_{i=1}^n$
Intersecting w/ A gives a finite cover of A:
$\{\emptyset\} \cup \{U_\alpha\}_{i=1}^n$. So the collection of U_{α_i}'s is a finite subcover.
$\therefore$ A is compact

(note, the converse is not true in general)
i.e. if A is a compact subset of X
need not be a closed subset of X.
BUT if X is Hausdorff, this is true

Consequences of the Thm.:

(12.) Corollary if X is a closed, bounded subset of $\mathbb{R}^n$, then X is compact.

Pf: Since X is bounded, X lies in some $\prod_{i=1}^{n} [-M, M] \subseteq \mathbb{R}^n$

Each $[-M, M]$ is compact b/c it is a closed interval.

$\therefore \prod_{i=1}^{n} [-M, M]$ is compact (product of compact sets) and X is a closed subset, so X is compact by the Thm.

(Counter Example of converse (14.) $\mathbb{Q} \cap [0, 1]$

this is a closed, bounded subset of $\mathbb{Q}$, but is not compact

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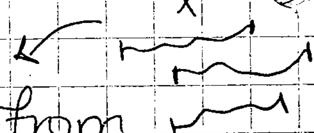
$\boxed{\text{April 12}}$ Connectedness: Let $\{1, 2\}$ be the space w/ two points 1 and 2 and has the discrete topology.

X



X is disconnected if $\overset{\cdot}{1}$ $\overset{\cdot}{2}$ there is a continuous surjection from X to $\{1, 2\}$

• X is connected if there is not cont. surjection from X to $\{1, 2\}$ (i.e. every cont. function from $\{1, 2\}$ is constant)

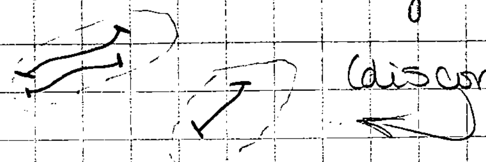
Proposition: The following are equivalent

(a) X is connected

(b) If A is a non-empty open & closed subset of X, then $A = X$

(X & ϕ are the only sets that are open & closed)

(c) X cannot be written as $U \cup V$ where U and V are disjoint, nonempty open sets



(disconnected)

(contrapositives : $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$)

Pf: • (a) ⇒ (b) : Argue $\neg b \Rightarrow \neg a$

Let A be a nonempty open & closed subset w/ $A \neq X$. (need to show X not conn.)

define $f: X \to \{1, 2\}$ by $f(x) = \begin{cases} 1 & \text{if } x \in A \\ 2 & \text{if } x \in X - A \end{cases}$

f is continuous b/c

$f^{-1}(\{1\}) = A$ (open) & $f^{-1}(\{2\}) = X - A$ (open)

f is surjective since A is not empty, & $X \neq A$, so $X - A$ is not empty

∴ X is disconnected

• (b) ⇒ (c) : Argue contrapositive

Suppose X can be written $U \cup V$ with U, V disjoint nonempty open sets.

U is open and U is closed (since $X - U = V$ is open)

V is open and V is closed (since $X - V = U$ is open)

∴ $U \neq \emptyset$ $U \neq X$ since $V \neq \emptyset$

• (c) ⇒ (a) : Argue contrapositive

Assume X is disconnected.

Let $f: X \to \{1, 2\}$ be a cont. surjection.

Let $U = f^{-1}(\{1\})$ and $V = f^{-1}(\{2\})$

U and V are disjoint and $U \cup V = X$

U and V are open since f is continuous and U & V are nonempty b/c f is surjective

$\boxed{\text{Ex}}$ $\mathbb{Q}$ is not connected

Pf: $\left(\mathbb{Q} = \overbrace{\mathbb{Q} \cap (-\infty, \sqrt{2})}^{U}\right) \cup \overbrace{\left(\mathbb{Q} \cap (\sqrt{2}, \infty)\right)}^{V}$

two disjoint nonempty sets

• however, <u>$\mathbb{R}$ is connected</u>

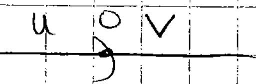
($\mathbb{R}$ is one solid piece)

(because $\mathbb{R}$ are complete, where as $\mathbb{Q}$ is not)

Aubrey

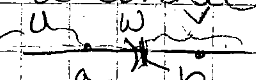
<u>pf</u>: Suppose $\mathbb{R}$ is disconnected.
let $f: \mathbb{R} \to \{1,2\}$ be a continuous surjection
The inclusion $i: \{1,2\} \to (\mathbb{R}, \mathcal{L})$ is continuous
$\therefore$ $i \circ f: \mathbb{R} \to (\mathbb{R}, \mathcal{L})$ is a comp. of cont. functions and nonconstant b/c f is surjective
This contradicts the fact that the only continuous functions from $\mathbb{R}$ to $(\mathbb{R}, \mathcal{L})$ are constant functions ■

$\boxed{\text{EX}}$ $(\mathbb{R}, \mathcal{L})$ is not connected



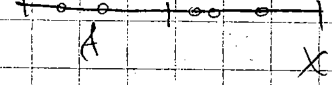
<u>pf</u>: $(\mathbb{R}, \mathcal{L}) = \underbrace{(-\infty, 0)}_{\text{open}} \cup \underbrace{[0, \infty)}_{\text{open}}$

<u>Refinement</u>:
if C is a connected subset of $(\mathbb{R}, \mathcal{L})$ then C consists of only one point
<u>pf</u>: Suppose for contradiction that C is a connected subset of $(\mathbb{R}, \mathcal{L})$ and $a, b \in C$ w/ $a < b$



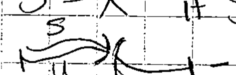
choose w with $a < w < b$
let $U = C \cap (-\infty, w)$ and
$V = C \cap [w, \infty)$
U and V are disjoint and nonempty b/c $a \in U$ and $b \in V$. They are open in C & their union is C.
This contradicts the fact that C is connected.
NOTE: also true in $\mathbb{Q}$
(homework)

(51.)



X cofinite

(54.) $S \subseteq X$ if S is connected, then $\overline{S}$ is connected



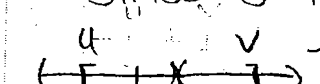
if can break into U & V, then reach a contradiction
$S \subseteq U$ must be in one of them, U open $\Rightarrow$ V closed
$\hookrightarrow \overline{S} \subseteq U$
Suppose $\overline{S} = U \cup V$ disjoint & nonempty (prove 1 is empty)
$S \subseteq U$ or $S \subseteq V$. Show if $S \subseteq U$, $\overline{S} \subseteq U \Rightarrow V$ empty $\times$

$\boxed{\text{April 14}}$ <u>Thm</u>: if J is <u>any interval</u> in $\mathbb{R}$, then J is connected: (J can be closed, open, or ½ open, could be a ray, $(-\infty,\infty)$, etc.)

<u>Pf</u>: Suppose for contradiction $J = U \cup V$ where U and V are disjoint & nonempty open subsets

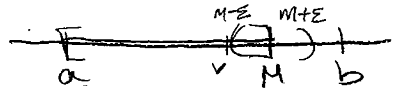
choose $a \in U$ and $b \in V$ & assume $a<b$

Since J is an interval, $a,b \in J$ $[a,b] \subseteq J$



let $S = \{x \in [a,b] \mid [a,x] \subseteq U\}$

let M = the least upper bound of S



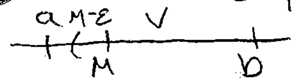
<u>Case I</u>: $M \in U$, then $M \neq b$ b/c $b \in V$

Claim $[a,M] \subseteq U$ (for if there were a $v \in V$ w/ $a<v<M$, then v would be a smaller upper bound for S)

Since U is open, there is some $[M, M+\varepsilon) \subseteq U \cap [a,b]$ for som $\varepsilon > 0$. $[a,M] \cup [M, M+\frac{\varepsilon}{2}] \subseteq U$,

So $M+\frac{\varepsilon}{2} \in S$, but M is an upper bound for S. $\times$

<u>Case II</u>: Suppose $M \in V$, Then $M \neq a$ since $a \in U$



Since V is open, $(M-\varepsilon, M] \subseteq V \cap [a,b]$

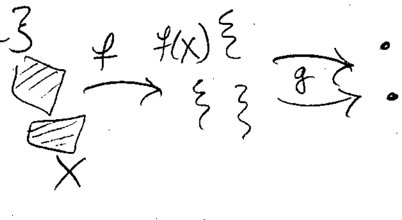
Then $M-\frac{\varepsilon}{2}$ is a smaller upper bound for S. $\times$

$\therefore$ M can't be in U or V

<u>Proposition</u>: if $f: X \to Y$ (cont.) and X is connected, then $f(X)$ is connected.

(contra-positive) <u>Pf</u>: Suppose $f(X)$ is not connected. Then there exists a cont. surjection $G: f(X) \to \{1,2\}$

But the $g \circ f: X \to \{1,2\}$ is a continuous surjection also. So X is not connected. ∎



$\boxed{\text{Ex}}$ $f: [0,2\pi] \to \mathbb{R}^2$ by $f(t) = (\cos(t), \sin(t))$ is a continuous surjection,

So $f([0,2\pi]) = S^1$ is connected

$\boxed{\text{Ex}}$: Every continuous function from any connected space to $(\mathbb{R}, \mathcal{L})$ or to $\mathbb{Q}$ is <u>constant</u>.

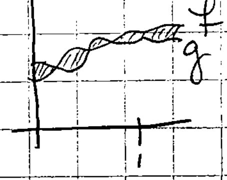
<u>Pf</u>: let X be connected and $f: X \to (\mathbb{R}, \mathcal{L})$ or $f: X \to \mathbb{Q}$ be continuous.
$f(X)$ is a connected set. In $(\mathbb{R}, \mathcal{L})$ or in $\mathbb{Q}$, the only connected subsets are single points, So f is constant.

<u>Metric Spaces</u>
- A metric is a concept of distance between points of a <u>set</u> X.
- A metric on X determines a topology on X, called the metric topology (for that metric)
- Not every topology is a metric topology — such topologies have a lot of nice properties

A <u>metric</u> on a set X is a function $d: X \times X \to \mathbb{R}$ satisfying
1. $\forall x, y$, $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$
2. $\forall x, y$, $d(x,y) = d(y,x)$
3. The triangle inequality, $\forall x, y, z$ $d(x,y) \leq d(x,z) + d(z,y)$

$\boxed{\text{Ex}}$ 1.) on $\mathbb{R}$, $d(x,y) = |x-y|$
on $\mathbb{R}^n$, $d(x,y) = \|x-y\|$
2.) on any subset X of $\mathbb{R}^n$, $d(x,y) = \|x-y\|$
If X has a metric d, and $A \subseteq X$, then $d|_{A \times A}$ is a metric on A.

3.) $X = \mathcal{C}([0,1], \mathbb{R})$ = set of all continuous functions from $[0,1]$ to $\mathbb{R}$
$d(f,g) = \int^1 |f(x) - g(x)| dx$
(the area between them is the distance)



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Ex ③ $X = \mathcal{C}([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ is cont.}\}$

$\rho(f,g) = \int_0^1 |f(t) - g(t)|\,dt$

metric?

① $\rho(f,g) \geq 0$. For $P(t) \geq 0$, $\int_0^1 P(t)\,dt = 0 \iff P = 0$

$\therefore \rho(f,g) = 0 \iff \int_0^1 |f(t) - g(t)|\,dt = 0$

$\iff f(t) = g(t) \quad 0 \leq t \leq 1$

$\iff f = g$

② $\rho(f,g) = \rho(g,f)$

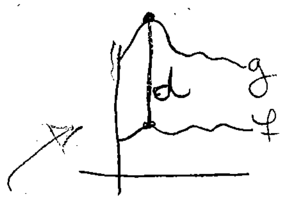
③ given f, g, h

$\rho(f,g) = \int_0^1 |f(t) - g(t)|\,dt$

$= \int_0^1 |f(t) - h(t) + h(t) - g(t)|\,dt$

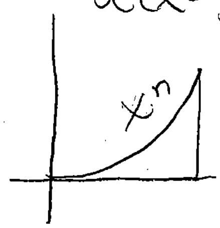
$\leq \int_0^1 |f(t) - h(t)|\,dt + \int_0^1 |h(t) - g(t)|\,dt$

$= \rho(f,h) + \rho(h,g)$



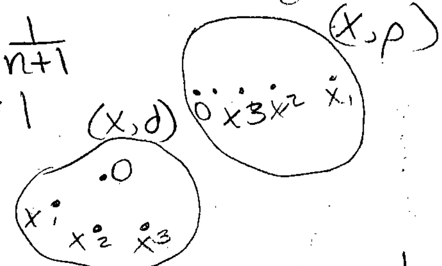
Ex ④ $X = \mathcal{C}([0,1]) \quad d(f,g) = \max |f(x) - g(x)|$ w/ $x \in [0,1]$

$d(x^3, 0) = 1$

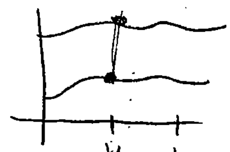


$\rho(x^n, 0) = \frac{1}{n+1}$

$d(x^n, 0) = 1$



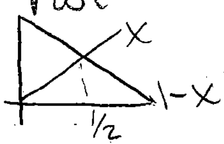
Ex ⑤ $X = \mathcal{C}([0,1]) \quad \tau(f,g) = |f(\tfrac{1}{2}) - g(\tfrac{1}{2})|$



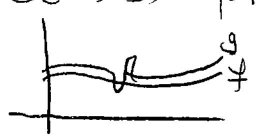
① of metric properties not satisfied

for ex., $\tau(x, 1-x) = 0$

(② & ③ are satisfied)



Ex ⑥ $X = \mathcal{C}([0,1]) \quad \rho_1(f,g) = \max\limits_{t \in [0,1]} \{|f(t) - g(t)| + |f'(t) - g'(t)|\}$



$\rho(f,g) \approx 0$

$\rho_1(f,g)$ is large

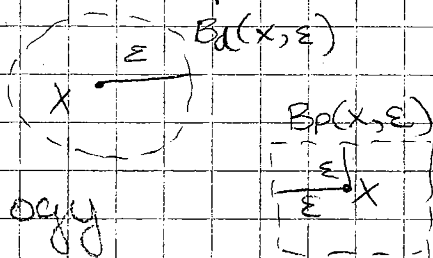
def: Let X be a set w/ metric d
for $x \in X$, $\varepsilon > 0$, define
$B(x,\varepsilon) = \{z \in X \mid d(z,x) < \varepsilon\}$
let $\mathcal{B} = \{B(x,\varepsilon) \mid x \in X, \varepsilon > 0\}$
verify $\mathcal{B}$ is a basis is exactly like for $\mathbb{R}^n$

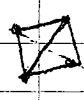
- The topology this generates is a metric topology for that metric, d

- If Y is a topological space & its topology is a metric topology for some metric, then Y is called metrizable

(which a lot of nice (easy to work w/) properties

Ex. On $X = \mathbb{R}^n$, the metrics
$\|x-y\| = d(x,y)$
$\max\{|x_i - y_i|\}_{i=1}^n = \rho(x,y)$
generate the standard topology

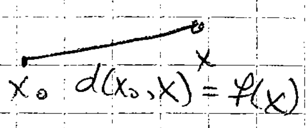


Ex. X = any set
define $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y \end{cases}$ (four points make  & 3 make 

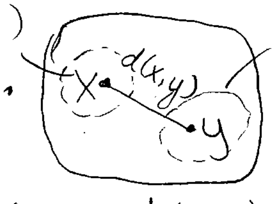
triangle ineq: $d(x,y) \le d(x,z) + d(x,y)$
Case I: $d(x,z) + d(z,y) = 0$, then $x = z$ & $z = y$,
so $d(x,y) = 0$
Case II: $d(x,z) + d(z,y) \neq 0$, $d(x,z) + d(z,y) \geq 1$
so $d(x,y) \le 1$
For every x, $B_d(x,\frac{1}{2}) = \{x\}$, so each point is an open set in the metric topology.
So the d-metric topology is the discrete topology

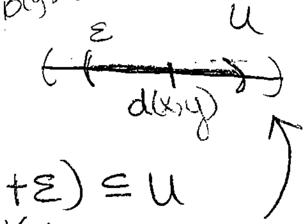
Proposition: Let (X,d) be a metric space
Then $d: X \times X \to \mathbb{R}$ is continuous

Usually fix one point, and think of

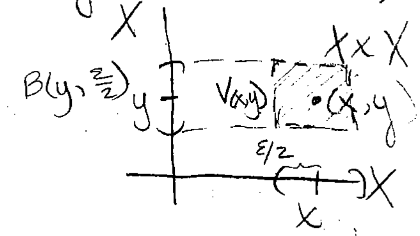


Pf: let U be open in $\mathbb{R}$.
Let $(x,y) \in d^{-1}(U)$
$d(x,y) \in U$
Choose $\varepsilon > 0$ with $(d(x,y)-\varepsilon, d(x,y)+\varepsilon) \subseteq U$





Put $V_{(x,y)} = B(x,\frac{\varepsilon}{2}) \times B(y,\frac{\varepsilon}{2})$,
a neighborhood of (x,y) in $X \times X$
(show this is in $d^{-1}(U)$)



If $(x',y') \in V(x,y)$, then
$$d(x',y') \le d(x',x) + d(x,y') \le d(x',x) + d(x,y) + d(y,y') < \frac{\varepsilon}{2} + d(x,y) + \frac{\varepsilon}{2} = d(x,y) + \varepsilon$$
$$d(x,y) \le d(x,x') + d(x',y') + d(y',y) \le \frac{\varepsilon}{2} + d(x',y') + \frac{\varepsilon}{2} < d(x',y') + \varepsilon$$
$$\Rightarrow d(x,y) - \varepsilon < d(x',y')$$
So $d(x',y') \in U$
$\therefore d^{-1}(U)$ is open. ■

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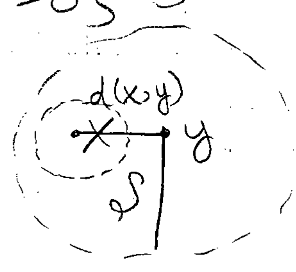
Handy fact: let (X,d) be a metric space.
A set U is open in the metric topology
$\Leftrightarrow \forall x \in U, \exists \varepsilon > 0, B(x,\varepsilon) \subseteq U$

Pf: "$\Leftarrow$" This condition implies U is open since a basis for the metric topology is a set of all $\mathcal{B} = \{B(y,\delta) \mid y \in X, \delta > 0\}$

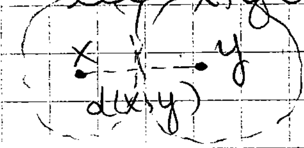
"$\Rightarrow$" Suppose U is open. Let $x \in U$
$\exists y \in X, \delta > 0$ so that $x \in B(y,\delta) \subseteq U$
(triangle inequality)
$$x \in B(x, \delta - d(x,y)) \subseteq B(y,\delta) \subseteq U$$



$\boxed{\text{Ex}}$ If X is an infinite set, then $(X, \text{cofinite})$ is not metrizable

Pf: Lemma: metrizable spaces are Hausdorff

- let Y be metrizable, so Y has the metric topology for some metric d
- let $x, y \in Y$, $x \neq y$



Claim $B(x, \frac{d(x,y)}{2})$ and $B(y, \frac{d(x,y)}{2})$ are disjoint neighborhoods of x & y

Suppose for contradiction that some z is in both balls. then $d(x,y) \leq d(x,z) + d(z,y)$

$$< \frac{d(x,y)}{2} + \frac{d(x,y)}{2} = d(x,y)$$

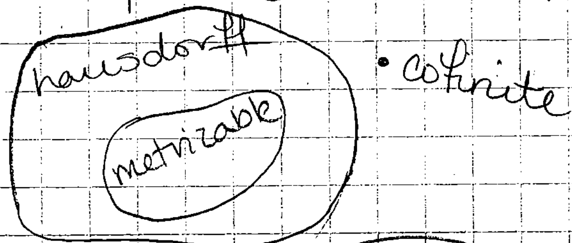
which is a contradiction. ∎

But if X is infinite, then any two nonempty open sets intersect:

Suppose $U = X - F_U$ and $V = X - F_V$ are open, so F_U and F_V are finite

$U \cap V = X - (F_U \cup F_V) \neq \emptyset$

So no two points have disjoint neighborhoods



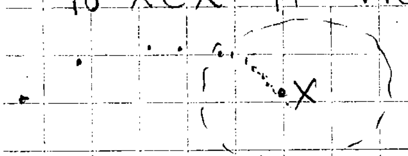
Sequences and limit points

def: let X be a topological space

let $\{x_n\}_{n=1}^{\infty}$ a sequence of points in X

(i.e. a function $f: \mathbb{N} \to X$, x_n denotes $f(n)$)

We say a sequence $\{x_n\}$ converges to $x \in X$ if $\forall$ nbd U of x, $\exists N$, $n \geq N \Rightarrow x_n \in U$



We can write $\lim x_n = x$ or $\{x_n\} \to x$