$$\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)} = \frac{f(x)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)}$$

$$\frac{(f(x)-f(x_0))g(x_0) + f(x_0)(g(x_0)-g(x))}{g(x)g(x_0)}$$

close to $g(x_0)$ when $x$ is close to $x_0$

HOMEWORK

## II. Euclidean Space

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}\} \qquad \mathbb{R}^1 = \mathbb{R}$$

$\mathbb{R}$ $\qquad$ $\mathbb{R}^2$ $\qquad$ $\mathbb{R}^3$

Induct

for $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$, define

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2 \dots + x_n^2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

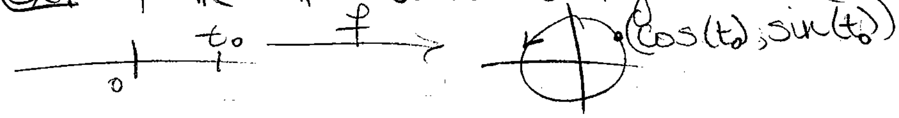
- This is the **norm** of $x$, the distance from $x$ to $(0,0,\dots 0)$
- We define the distance from $x$ to $y$ by the norm of $\|x-y\|$
- In $\mathbb{R}^1$, $\|x\| = (x^2)^{1/2} = |x|$ — (the norm generalizes this)

$$\|x-y\| = \|(x_1 \cdots x_n) - (y_1 \cdots y_n)\| = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$$

- $\mathbb{R}^n$ with this choice of the distance function is called n-dimensional Euclidean Space.

Let $f: D \to \mathbb{R}^n$ where $D \subseteq \mathbb{R}^m$
Let $x_0 \in D$
if $f$ is <u>continuous</u> at $x_0$ when
$\forall \varepsilon > 0, \exists \delta > 0$, $x \in D$ and $\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon$
(of course $f$ is cont. if it is cont. @ every $x_0 \in D$)

$\boxed{\text{Ex}}$ $f: \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (\cos(t), \sin(t))$



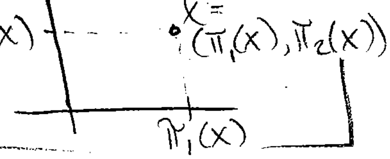

$f$ is cont.: let $t_0 \in \mathbb{R}$, let $\varepsilon > 0$.
There is a $\delta_1 > 0$ so that if
$|t - t_0| < \delta_1$, then $|\cos(t) - \cos(t_0)| < \frac{\varepsilon}{2}$
(Since cosine is continuous)

There is a $\delta_2 > 0$ so that if $|t - t_0| < \delta_2$, then
$|\sin(t) - \sin(t_0)| < \frac{\varepsilon}{2}$
(Since sine is continuous)

So, if $|t - t_0| < \min\{\delta_1, \delta_2\}$, then $\|f(t) - f(t_0)\| =$
$\|\cos(t) - \cos(t_0), \sin(t) - \sin t_0\| =$
$= ((\cos(t) - \cos(t_0))^2 - (\sin(t) - \sin(t_0))^2)^{1/2}$
$(|\cos(t) - \cos(t_0)|^2 - |\sin(t) - \sin(t_0)|^2)^{1/2} < \left(\left(\frac{\varepsilon}{2}\right)^2 + \left(\frac{\varepsilon}{2}\right)^2\right)^{1/2}$
$= \left(\frac{\varepsilon^2}{2}\right)^{1/2} = \frac{|\varepsilon|}{\sqrt{2}} < \varepsilon$. ■

$g: \mathbb{R} \to \mathbb{R}^2$ $g(t) = (g_1(t), g_2(t))$ $x = g_1(t)$ $y = g_2(t)$
Notice that this argument would work for any two continuous functions in the role of cosine & sine

For $1 \leq k \leq n$. $\pi_k: \mathbb{R}^n \to \mathbb{R}$ denotes the <u>projection function</u> defined by
$\pi_k((x_1, x_2, \ldots, x_n)) = x_k$



let $f: \mathbb{R}^m \to \mathbb{R}^n$ can be written as
for $x \in \mathbb{R}^m$, $f(x) = (\pi_1 f(x), \pi_2 f(x), \dots, \pi_n f(x))$

Write $f_k: \mathbb{R}^m \to \mathbb{R}$
for the function $\pi_k \circ f$

This is called the <u>$k^{th}$ coordinate function of $f$</u>

Then, $f(x) = (f_1(x), f_2(x), \dots f_n(x))$
An $f$ from $f: \mathbb{R}^m \to \mathbb{R}^n$ determines
n coordinate functions $f_k: \mathbb{R}^m \to \mathbb{R}$.

On the other hand, if we start with n functions,
$g_1, \dots, g_n : \mathbb{R}^m \to \mathbb{R}$, we can define a
$g: \mathbb{R}^m \to \mathbb{R}^n$ by $g(x) = (g_1(x), g_2(x), \dots, g_n(x))$

$\boxed{Ex}$ $f: \underset{\mathbb{C}}{\overset{\shortparallel}{\mathbb{R}^2}} \to \underset{\mathbb{C}}{\overset{\shortparallel}{\mathbb{R}^2}}$ $(x,y) = x + iy$
defined by $f(z) = z^2$

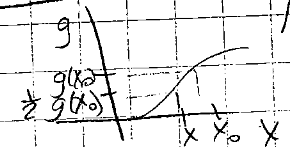
Coordinate functions of $f$:

$$f(x+iy) = (x+iy)^2 = x^2 + 2xiy + (iy)^2$$
$$= x^2 - y^2 + i \cdot 2xy$$

$$f(x+iy) = (x^2 - y^2, 2xy)$$
$$f_1(x,y) = x^2 - y^2 \text{ and } f_2(x,y) = 2xy$$

HOMEWORK

$\overline{g(x)g(x_0)}$ $g(x_0) \neq 0$



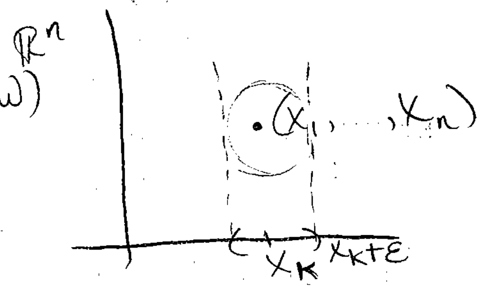
So $\exists \delta$, if
$|x - x_0| < \delta$, then $|g(x) - g(x_0)| < \frac{1}{2}|g(x_0)|$, in which case

$$|g(x_0)| = |g(x_0) - g(x) + g(x)| \le |g(x_0) - g(x)| + |g(x)|$$
$$\le \frac{1}{2}|g(x_0)| + |g(x)|$$
$$|g(x)| \ge \frac{1}{2} g(x_0)$$

$\pi_K : \mathbb{R}^n \to \mathbb{R} \qquad \pi_K((x_1, x_n)) = x_K$

$f: \mathbb{R}^m \to \mathbb{R}^n \quad \underbrace{f(x_1,\ldots,x_m)}_{\in \mathbb{R}^m} = \underbrace{(\pi_1 \circ f)}_{f_1}(x_1,\ldots,x_m), \underbrace{(\pi_2 \circ f)}_{f_2}(x_1,\ldots,x_m)$

$f_i : \mathbb{R}^m \to \mathbb{R}$




- Each $\pi_K$ is continuous (HW)

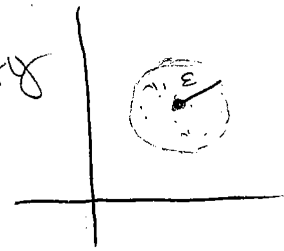
So given $\varepsilon$, choosing $\delta = \varepsilon$ should work.

- Consequently, if $f: \mathbb{R}^m \to \mathbb{R}^n$ is continuous, then each coordinate function $f_i = \pi_i \circ f$ is a composition of cont. functions (HW)
- Conversely, if each $f_i$ is continuous, then $f$ is continuous

$\boxed{\text{Ex}}$ $\mathbb{R} \to \mathbb{R}^2$, $t \to (\cos t, \sin(t))$ is continuous because cosine & sine are continuous

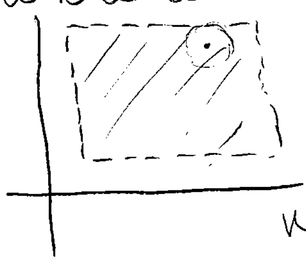
## Open Balls & Open Sets

NOTE: doesn't include boundary




def: let $x$ be a point in $\mathbb{R}^n$, let $\varepsilon > 0$
define $B(x, \varepsilon) = \{ z \in \mathbb{R}^n \mid \|z - x\| < \varepsilon \}$
This is the Open Ball with radius $\varepsilon$ & center @ $x$
type of object? – a set

Proposition let $W \subseteq \mathbb{R}^n$. Then $\forall x \in W, \exists \varepsilon > 0, B(x, \varepsilon) \subseteq W \Leftrightarrow$
$W$ is a union of open balls.
infinitely many



NOTE: Not possible for a closed square

no $\varepsilon$ for this $x$.

NOTE: Also not possible if $W$ is a subset of $\mathbb{R}^n$

"$\Longrightarrow$"

Pf: Suppose $\forall x \in W, \exists \varepsilon > 0, B(x,\varepsilon) \subseteq W$

$\forall x \in W$, choose a specific value, $\varepsilon_x > 0$ so that the ball around $x$ with radius $\varepsilon$, so that $B(x, \varepsilon_x) \subseteq W$

We will show that $W = \bigcup_{x \in W} B(x, \varepsilon_x)$

Suppose $z \in W$ and $z \in B(z, e_z)$

$\therefore z \in \bigcup_{x \in W} B(x, e_x)$

Suppose $z \in \bigcup_{x \in W} B(x, e_x)$

Then $z \in B(y, \varepsilon_y)$ for some $y \in W$.

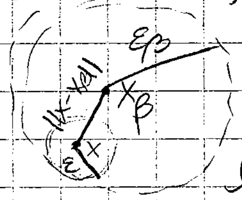
$B(y, \varepsilon_y) \subseteq W$, $\therefore z \in W$

Better way: $W = \bigcup_{x \in W} \{x\} \overset{\text{since each } \{x\} \subseteq B(x,\varepsilon_x)}{\subseteq} \bigcup_{x \in W} B(x, e_x) \overset{\text{since } B(x, e_x) \subseteq W}{\subseteq} W$

"$\Longleftarrow$"

Conversely, Suppose $W$ is a union of open balls, say $W = \bigcup_{\alpha \in a} B(x_\alpha, \varepsilon_\alpha)$

Let $x \in W$, Then $x \in B(x_\beta, \varepsilon_\beta)$ for some $\beta \in a$




put $\varepsilon = \varepsilon_\beta - \|x - x_\beta\|$

$\varepsilon > 0$ since $x \in B(x_\beta, \varepsilon_\beta)$ implies $\|x - x_\beta\| < \varepsilon_\beta$

let $z$ be in $B(x, \varepsilon)$ (so $\|z - x\| < \varepsilon - \|x - x_\beta\|$)

$$\|z - x_\beta\| = \|z - x + x - x_\beta\|$$
$$\leq \|z - x\| + \|x - x_\beta\|$$
$$\leq \underbrace{(\varepsilon_\beta - \|x - x_\beta\|)}_{\varepsilon} + \|x - x_\beta\| = \varepsilon_\beta$$

$\therefore z \in B(x_\beta, \varepsilon_\beta)$ $\therefore z \in W$

$\therefore B(x, \varepsilon) \subseteq W$.

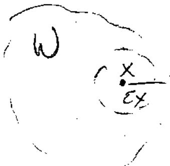
Review

$W \subseteq \mathbb{R}^n$

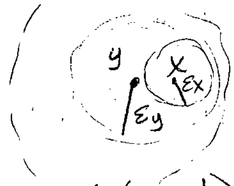
$W$ is a union of open balls

$\iff \forall x \in W, \exists \varepsilon > 0, B(x, \varepsilon) \subseteq W$

"$\Leftarrow$"



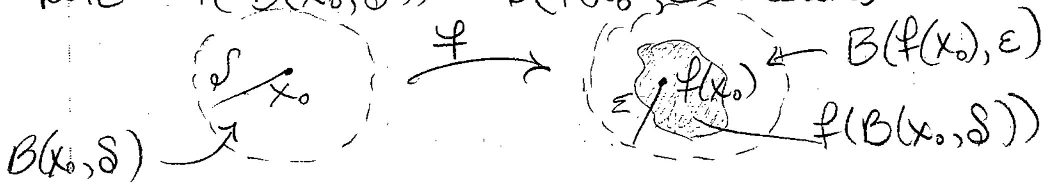

"$\Rightarrow$"




$d(a,b) \leq d(a,c) + d(c,d)$

$\|a-b\| \leq \|a-c+c-b\| \leq \|a-c\| + \|c-b\|$

NOTE: $f(B(x_0,\delta)) \subseteq B(f(x_0),\varepsilon)$ means




Exactly the same thing

in the open ball when

$\|x-x_0\| < \delta$, then $\|f(x)-f(x_0)\| < \varepsilon$

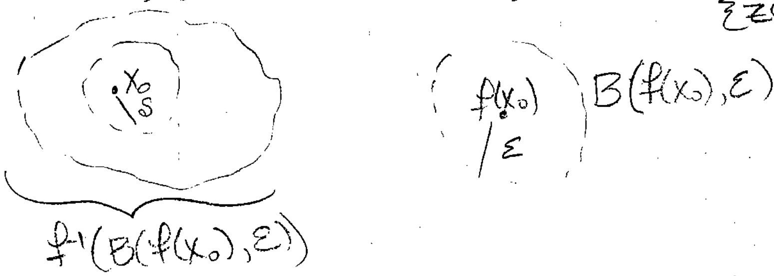
So, $f: \mathbb{R}^m \to \mathbb{R}^n$ is cont. @ $x_0$ when

$\forall \varepsilon > 0, \exists \delta > 0, f(B(x_0,\delta)) \subseteq B(f(x_0),\varepsilon)$

Notice also that $f(B(x_0,\delta)) \subseteq B(f(x_0),\varepsilon)$

is the same as saying

$B(x_0,\delta) \subseteq f^{-1}(B(f(x_0),\varepsilon))$ ← preimage = set of all points $\{z \in \mathbb{R}^m \mid f(z) \in B(f(x_0),\varepsilon)\}$




<u>def</u>: A subset $U \subseteq \mathbb{R}^n$ is called <u>open</u> when

$\forall x \in U, \exists \varepsilon > 0, B(x,\varepsilon) \subseteq U$

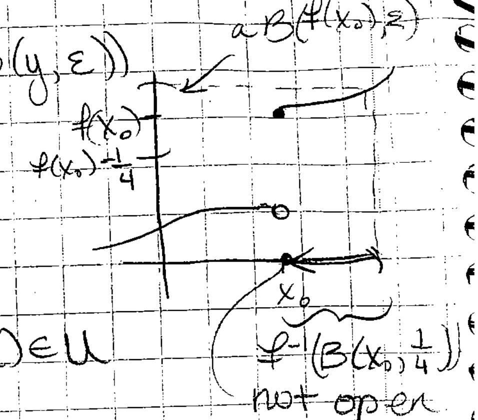
or equivalently, $U$ is a union of open balls

**Thm.** $f:\mathbb{R}^m \to \mathbb{R}^n$ the following 3 things are equivalent:

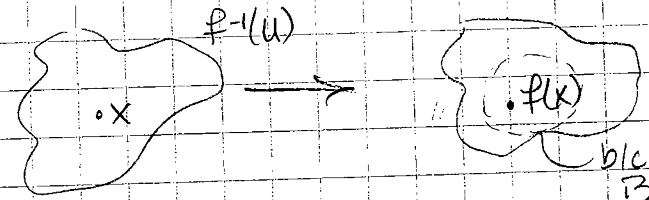
① $\forall x_0 \in \mathbb{R}^m$ and $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t.
$\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon$

② for every open set $U \subseteq \mathbb{R}^n$, $f^{-1}(U)$ is open in $\mathbb{R}^m$

③ for every open ball $B(y,\varepsilon)$ in $\mathbb{R}^n$, $f^{-1}(B(y,\varepsilon))$ is open in $\mathbb{R}^m$




Pf: ① ⇒ ②
Assume ① is True,
let $U$ be an open set in $\mathbb{R}^m$
let $x$ be an element of $f^{-1}(U)$, so $f(x) \in U$




b/c $U$ is open, $\exists \varepsilon > 0$ s.t. $B(f(x),\varepsilon) \subseteq U$

by ①, $\exists \delta > 0$ s.t. $f(B(x,\delta)) \subseteq B(f(x),\varepsilon) \subseteq U$
i.e. $B(x,\delta) \subseteq f^{-1}(B(f(x)),\varepsilon) \subseteq f^{-1}(U)$
$\therefore f^{-1}(U)$ is open

② ⇒ ③ Let $B(y,\varepsilon)$ be an open ball.
$B(y,\varepsilon)$ is open (since it's a union of 1 open balls),
so by ②, $f^{-1}(U)$ is open.

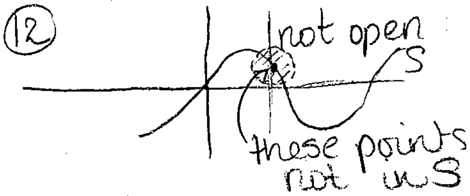
③ ⇒ ① Assume ③, let $x_0 \in \mathbb{R}^m$ and let $\varepsilon$ be given.
$f(x_0) \in B(f(x_0),\varepsilon)$, so $x_0 \in f^{-1}(B(f(x_0),\varepsilon))$.
By ③ this is open $\exists \delta > 0$ s.t. $B(x_0,\delta) \subseteq f^{-1}(B(f(x_0),\varepsilon))$
$f(B(x_0,\delta)) \subseteq B(f(x_0),\varepsilon)$.
If $\|x - x_0\| < \delta$, then $\|f(x) - f(x_0)\| < \varepsilon$. ■

$\omega$ is open if $\forall x \in \omega, \exists \varepsilon > 0,$
$B(x,\varepsilon) \subseteq \omega$

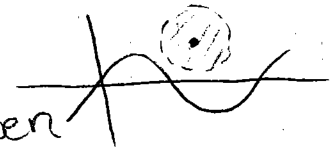
$\mathbb{Q} \subseteq \mathbb{R}$
not open $\quad$ $p/q-\varepsilon \quad p/q \quad p/q+\varepsilon$
$\hookrightarrow$ contains an irrational, so
$(p/q-\varepsilon,\ p/q+\varepsilon) \not\subseteq \mathbb{Q}$ $\quad$ $r-\varepsilon \quad r \quad r+\varepsilon$ contains a rational

- $\mathbb{R}-\mathbb{Q}$ is also not open for the same reason

(12)




- no open balls fit inside the graph



HW

but the complement, $\mathbb{R}^2 - S$ is open

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$f: \mathbb{R}^m \to \mathbb{R}^n$ is continuous (in the $\varepsilon$-$\delta$ def.)
$\iff \forall U$ open in $\mathbb{R}^n$, $f^{-1}(U)$ is open in $\mathbb{R}^m$

<u>Properties of Open Sets in $\mathbb{R}^n$</u>

(1.) Let $\{U_\alpha\}_{\alpha \in a}$ be a collection of open sets in $\mathbb{R}^n$. Then $\bigcup_{\alpha \in a} U_\alpha$ is open.

<u>Pf.</u>: let $x \in \bigcup U_\alpha$. Then $x \in U_\beta$ for some $\beta \in a$
$U_\beta$ is open, so $\exists \varepsilon > 0, B(x,\varepsilon) \subseteq U_\beta \subseteq \bigcup U_\alpha$. ■

(2.) Let $\{U_1, U_2, \ldots, U_K\}$ be a finite collection of open subsets of $\mathbb{R}^n$. Then $\bigcap_{i=1}^{K} U_i$ is open

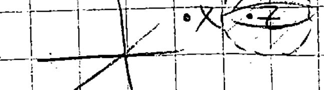
<u>Pf</u>: let $x \in \bigcap U_i$. For each $i$, $x \in U_i$ and $U_i$ is open, so $\exists \varepsilon_i$ s.t. $B(x,\varepsilon_i)$ is contained in $U_i$.
Let $\varepsilon = \min\{\varepsilon_1, \ldots \varepsilon_K\} > 0$. For each $i$, $\varepsilon \le \varepsilon_i$
so $B(x,\varepsilon) \subseteq B(x,\varepsilon_i)$
$\therefore B(x,\varepsilon) \subseteq \bigcap_{i=1}^{K} B(x,\varepsilon_i) \subseteq \bigcap_{i=1}^{K} U_i$ ■

if infinite collection of $\varepsilon_i$, there is no minimum

NOTE: (2.) can fail for infinitely many open sets.

<u>Ex1</u> $\underbrace{\bigcap_{i=1}^{\infty} \left(-\tfrac{1}{n}, \tfrac{1}{n}\right)}_{\text{open}} = \underbrace{\{0\}}_{\text{not open}}$ $\quad$ $-1 \quad -1/2 \quad -1/3 \quad 0 \quad 1/3 \quad 1/2$

Similarly, $\bigcap_{i=1}^{\infty} B(x, \tfrac{1}{n}) = \{x\}$



$\boxed{\text{Ex}}$ for all $x \in \mathbb{R}^n$, $\mathbb{R}^n - \{x\}$ is open

Pf: let $z \in \mathbb{R}^n - \{x\}$, i.e. $z \neq x$

$\|z-x\| > 0$, so $B(z, \frac{\|z-x\|}{2}) \subseteq \mathbb{R}^n - \{x\}$

(for if not, then it would have to contain $x$, so $\|x-z\| < \frac{\|z-x\|}{2}$

$\|x-z\| < 0$ for a contradition)

$\boxed{\text{Ex}}$ let $\{x_1, \ldots x_k\}$ be a finite collection of points in $\mathbb{R}^n$. Then $\mathbb{R}^n - \{x_1, \ldots, x_k\}$ is open.

Pf: $\mathbb{R}^n - \{x_1, \ldots, x_k\} = \mathbb{R}^n - (\bigcup_{i=1}^{k} \{x_i\})$

$= \bigcap_{i=1}^{k} (\mathbb{R}^n - \{x_i\})$ (De Morgan's law: if ea. $U_\alpha \subseteq W$, then $W - \bigcup U_\alpha = \bigcap (W - U_\alpha)$)

This is an intersection of finitely many open sets, so it is open.

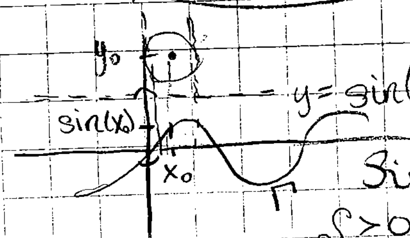
Note: Every subset of $\mathbb{R}^n$ is an intersection of some collection of open sets

Pf: Let $A \subseteq \mathbb{R}^n$. $\bigcap_{x \in \mathbb{R}^n - A} \underbrace{\mathbb{R}^n - \{x\}}_{\text{open}}$ (the collection of open sets is $\{\mathbb{R}^n - \{x\} \mid x \in \mathbb{R}^n - A\}$)

by De Morgan's law,

$= \mathbb{R}^n - \bigcup_{x \in \mathbb{R}^n - A} \{x\} = \mathbb{R}^n - (\mathbb{R}^n - A) = A$

(NOTE) $|x_i - y_i| \leq \|(x_1, x_2, \ldots x_n) - (y_1, y_2, \ldots y_n)\|$




Claim:

① $\mathbb{R}^2 - \Gamma$ is open

Pf: let $(x_0, y_0) \in \mathbb{R}^2 - \Gamma$

Since sin is continuous, there exists $\delta > 0$ s.t. if $|x - x_0| < \delta$,

then $|\sin(x) - \sin(x_0)| < \frac{|y_0 - \sin(x_0)|}{2}$

· radius must be smaller than $\delta$ and

Put $\varepsilon = \min\{\delta, |\frac{y_0 - \sin(x_0)}{2}|\}$

Now need to show the Ball is in $\mathbb{R}^2 - \Gamma$ — use! Contradiction

- Suppose $B((x_0, y_0), \varepsilon) \cap \Gamma \neq \emptyset$

Then $\exists x$ with $(x, \sin(x)) \in B((x_0, y_0), \varepsilon)$

$|x - x_0| \leq \|(x, \sin(x)) - (x_0, y_0)\| < \varepsilon \leq \delta$ b/c supposing in Ball

$\therefore |\sin(x) - \sin(x_0)| < \frac{|y_0 - \sin x_0|}{2}$

$$\|y_0 - \sin(x)\| \le \|(x, \sin(x)) - (x_0, y_0)\| < \varepsilon \le \frac{|y_0 - \sin(x_0)|}{2}$$

$$|y_0 - \sin(x_0)| \le |y_0 - \sin(x)| + |\sin(x) - \sin(x_0)| <$$

$$\frac{|y_0 - \sin(x_0)|}{2} + \frac{|y_0 - \sin(x_0)|}{2} = |y_0 - \sin(x_0)| \quad \times$$

Contradiction: The number is less than itself

$\therefore B((x_0, y_0), \varepsilon) \subseteq \mathbb{R}^2 - \Gamma$

② Pf: define $F: \mathbb{R}^2 \to \mathbb{R}$ by $F(x,y) = y - \sin(x) \leftarrow$ continuous

$$F(x,y) = 0 \iff y = \sin(x)$$
$$\iff (x,y) \in \Gamma$$

so $\Gamma = F^{-1}(\{0\})$, so $\mathbb{R}^2 - \Gamma = \underbrace{F^{-1}(\underbrace{\mathbb{R} - \{0\}}_{\text{open}})}_{\text{open}}$

open since $F$ is cont. ∎

NOTE: Can generalize both of these to show the graph of any cont. function $f: \mathbb{R} \to \mathbb{R}$ is cont.

---

If $g: X \to Y$ is a function, $X \times Y$ is the set of ordered pairs where $\{(x,y) \mid x \in X, y \in Y\}$

The graph of $g$ is $\Gamma_g = \{(x, g(x)) \mid x \in X\}$

Prop: if $g$ is a continuous function between topological spaces, then $X \times Y - \Gamma_g$ is always an open set

① Pf: generalizes to prove this, but ② Pf. doesn't becaus $x - g(x)$ doesn't make sense unless $Y$ has a subtraction operation

?: if $f: \mathbb{R} \to \mathbb{R}$ and $\mathbb{R}^2 - \Gamma_f$ is open, must $f$ be cont.?

motivation for def of Top. Space: (Basic Ideas)

continuity for maps $f: \mathbb{R}^m \to \mathbb{R}^n$
$\varepsilon$-$\delta$ definition
↓
reformed to minimize the role of distance (by thinking ab sets instead of distances)
For $x \in \mathbb{R}^k$, define $B(x, \varepsilon) = \{z \in \mathbb{R}^k \mid \|z - x\| < \varepsilon\}$
define $U \subseteq \mathbb{R}^k$ to be open if...
(equivalently, $U$ is open when $U$ is a union of open balls)
$f$ satisfies the $\varepsilon$-$\delta$ def. $\Leftrightarrow$ $\forall U$ open in $\mathbb{R}^n$, $f^{-1}(U)$ is open in $\mathbb{R}^m$

Properties of Open Sets in $\mathbb{R}^n$ ⇝ general def. of the open sets of a space

$f$ is cont. when $\forall U$ open, $f^{-1}(U)$ is open ⇝ gen. def. of cont. of $f: X \to Y$

Open set in $\mathbb{R}^n$ is a union of $\varepsilon$-balls ⇝ a "basis" of sets that "generate" the topology
- every open set is a union of basic open sets

Standard Topology on $\mathbb{R}^n$ derives from the topology on $\mathbb{R}$:
a basis is $\underbrace{\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)\}}_{\text{open rectangles on } \mathbb{R}^n}$ ⇝ general def. of "product topology" on $X \times Y$ when $X$ and $Y$ have topologies

Properties of distance in $\mathbb{R}^n$ ⇝ general concept of a metric $d$ (HW 10)
⇝ gives a metric topology — but not all topologies come from metrics