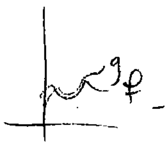
☆ Continuity & Compactness ☆

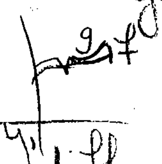
1st general def. of Continuity:

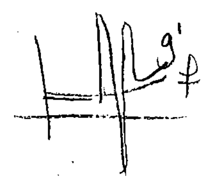
e.g. Is $\int_0^1 f(x)\,dx$ a [function associating a number to each (integrable) function f.] continuous function of f?



-seems like nearby functions have approximately the same integral

what ab d/dx? Perhaps not:





-eventhough similar numerically, the tangent lines are very diff

Continuity for regular func:

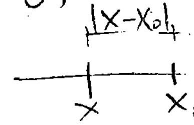
$f: \mathbb{R} \to \mathbb{R}$, or generally $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$

def: let $x_0 \in D$. f is continuous at x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, \ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

i.e. $\forall \varepsilon > 0, \exists \delta > 0$, if $x \in D$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$

note: $|x - x_0|$ is the distance b/two x to x_0



• if f is continuous, it is cont. at x_0 for all $x_0 \in D$

[Ex] Prove that $f: [0, \infty] \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is continuous

$\forall \varepsilon > 0, \exists \delta > 0$, if $x \geq 0$ and $|x - x_0| < \delta$, then $|\sqrt{x} - \sqrt{x_0}| < \varepsilon$

1.) Statement is of the form $\forall \varepsilon$, [ ]
so proof has form:
pf: let $\varepsilon > 0$

2.) $\exists \delta$ [ ] so, pf: let $\varepsilon > 0$ [ ] Put $\delta =$ [ ], [ ] ← verify δ has the property

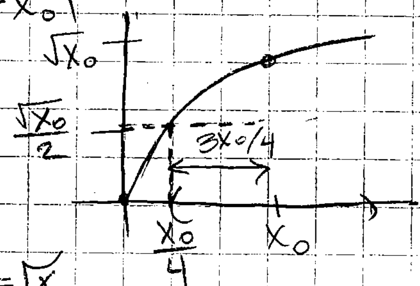
3. an implication: So expect a direct argument, $P \Rightarrow Q$
let $\varepsilon > 0$, put $\delta = \boxed{}$, Suppose $x \in D$ and $|x - x_0| < \delta$, (assuming P is true) $\boxed{}$ $\therefore |\sqrt{x} - \sqrt{x_0}| < \varepsilon$ (Q is implied)

4. Analyze: relate $|\sqrt{x} - \sqrt{x_0}|$ to $|x - x_0|$

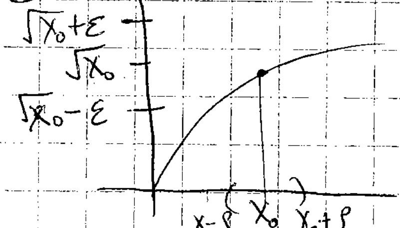
$$x - x_0 = (\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})$$

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{|\sqrt{x} + \sqrt{x_0}|}$$

if $|x - x_0| < \frac{3x_0}{4}$, then $\sqrt{x} > \frac{\sqrt{x_0}}{2}$



Continued: $f : [0, \infty) \to \mathbb{R}$ $f(x) = \sqrt{x}$
~~If~~ $\forall \varepsilon > 0, \exists \delta > 0$, if $x \in [0, \infty)$, then $|x - x_0| < \delta$, then $|\sqrt{x} - \sqrt{x_0}| < \varepsilon$

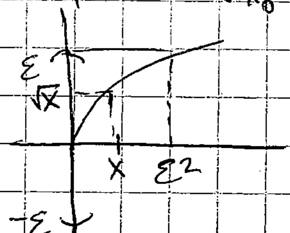


if $x_0 \neq 0$

$$|\sqrt{x} - \sqrt{x_0}| = \frac{x - x_0}{|\sqrt{x} + \sqrt{x_0}|} \leq \frac{|x - x_0|}{\sqrt{x_0}} < \frac{\varepsilon \sqrt{x_0}}{\sqrt{x_0}} = \varepsilon$$

provided $|x - x_0| < \varepsilon \cdot \sqrt{x_0}$

when $x_0 = 0$?



Pf: let $\varepsilon > 0$ be given

Case I: $x_0 \neq 0$ put $\delta = \varepsilon \sqrt{x_0}$. If $x \geq 0$ & $|x - x_0| < \delta$, then

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{|\sqrt{x} + \sqrt{x_0}|} \leq \frac{|x - x_0|}{\sqrt{x_0}} < \frac{\varepsilon \sqrt{x_0}}{\sqrt{x_0}} = \varepsilon$$

Case II: $x_0 = 0$ Put $\delta = \varepsilon^2$

if $x \geq 0$ and $|x - 0| < \delta$, then

$$|\sqrt{x} - \sqrt{0}| = \sqrt{x} \leq \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon \quad \blacksquare$$

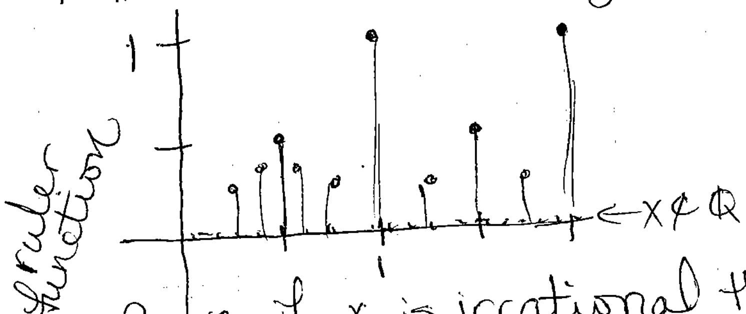
$D \subseteq \mathbb{R}$ $\quad f: D \to \mathbb{R}$, $x_0 \in D$

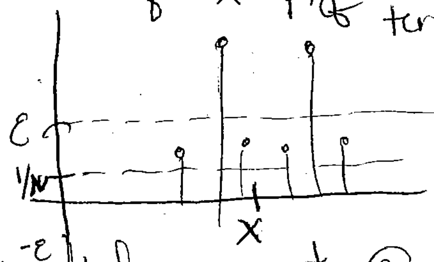
f is continuous @ x_0 if

for every $\varepsilon > 0$, $\exists\, \delta > 0$, if $x_0 \in D$, $|x_0 - x| < \delta$, then $|f(x_0) - f(x_0)| < \varepsilon$

Example 2: The "ruler" function

$f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/q & \text{if } x \text{ is rational } x = p/q \text{ in lowest terms } q > 0 \end{cases}$





• Prove if x is irrational then f is cont. @ x:

Pf: let $\varepsilon > 0$ be given, Choose a positive integer N with $1/N < \varepsilon$. Between $x-1$ and $x+1$ (i.e. for numbers z with $|z - x| < 1$) There are only finitely many $r_i = \frac{p_i}{q_i}$ with $q_i \le N$. say $r_1, \dots r_k$ and there is at least one such r_i (w/ $q_i = 1$)

put δ equal to min of $\{|r_1 - x|, |r_2 - x|, \dots |r_k - x|\}$

$\delta > 0$ b/c x is irrational (← using hyp.)

also, $\delta < 1$ since all r_i are b/tw $x-1$ and $x+1$.

→ Suppose $|z - x| < \delta$, ~~$z\,x$~~ $|z - x| < 1$ so z lies b/tw $|x-1|$ and $|x+1|$

also, z is not equal to any r_i since $|z - x| < \delta = \min\{|r_1 - x|\}$

- If z is $\notin \mathbb{Q}$ then $|f(z) - f(x)| = |0 - 0| = 0 < \varepsilon$
- If $z \in \mathbb{Q}$ then $q > N$ b/c if $q = N$, z would equal r_i so $z = p/q$ $\quad |f(z) - f(x)| = |1/q - 0| = \frac{1}{q} < \frac{1}{N} < \varepsilon$ ∎

Negation of Quantifiers

$\forall x \in S, P(x)$

$\forall x \in \mathbb{R}, x^2 \geq 0 \iff \exists x \in \mathbb{R}, x^2 < 0$

"not" $\neg \forall x \in S, P(x)$

$\forall x \in \mathbb{R}, x > 5$

$\exists x \in \mathbb{R}, x \leq 5 \iff \exists x \in S, \neg P(x)$

F

$\exists x \in \mathbb{R}, P(x) \quad \exists x \in \mathbb{R}, x^2 < 0 \quad \forall x \in \mathbb{R}, x^2 \geq 0$

$\exists x \in S, P(x) \ldots, \neg \exists x \in S, P(x) \iff \forall x, \neg P(x)$

$\neg \exists x, (P x) \iff \forall x, \neg P(x)$

$\neg \forall x, P(x) \iff \exists x \neg P(x)$

$\neg Q$ true

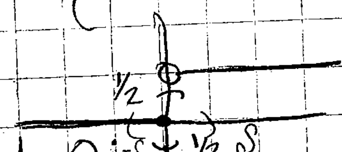
$\neg (P \Rightarrow Q) \iff P$ true, Q False

$\neg (P \Rightarrow Q) \iff P \wedge \neg Q$

Ex

$\neg (x < 4 \Rightarrow x < 2)$

$\iff (x < 4 \wedge x \geq 2)$



Ex define $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$

Prove that g is not continuous at 0.

Method I: Prove the negation of the def.

$\neg$(g is cont. @ zero)

$\neg(\forall \varepsilon > 0, \exists \delta > 0, \forall x, |x| < \delta \Rightarrow |g(x)| < \varepsilon)$

$\exists \varepsilon > 0, \neg($

$\exists \varepsilon > 0, \forall \delta > 0, \exists x, \neg(|x| < \delta \Rightarrow |g(x)| < \varepsilon)$

$\exists \varepsilon > 0, \forall \delta > 0, \exists x, |x| < \delta \wedge |g(x)| \geq \varepsilon$.

Proof: Put $\varepsilon = 1/2$, Let $\delta > 0$ be given.

Put $x = \frac{\delta}{2}$ $\quad |\frac{\delta}{2}| = \frac{\delta}{2} < \delta$, and $|g(\frac{\delta}{2})| = |1| = 1 \geq 1/2$. ∎

Method II: Proof by contradiction
P is true, assume it is false, show that it leads to an impossibility.

Proof: Suppose for contradiction that g is cont. at 0. Then if $\varepsilon = \frac{1}{2}$, there is a $\delta > 0$ so that $\forall x$ if $|x| < \delta$, then $|g(x)| < 1/2$

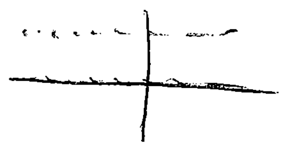
But $|\frac{\delta}{2}| < \delta$ and $|g(\frac{\delta}{2})| = 1 \nless 1/2 = \varepsilon$ ⨯

$\therefore g$ must be discontinuous @ zero

Note: Proof by Contradiction is useful for proving something doesn't exist

HW:

$g: \mathbb{R} \to \mathbb{R}$ $\quad g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$



discont. @ every point

- every interval of rat. contains inf. many irrat.

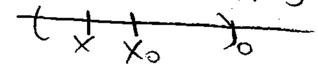
Ex let $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$, n a pos. int.
Prove that f is continuous

Method I: brute force

let ε be given and let $x_0 \in \mathbb{R}$

Case 1: $x_0 \neq 0$ put $\delta = \min\{|x_0|, \frac{\varepsilon}{2^n \cdot n \cdot |x_0|^{n-1}}\}$

if $|x - x_0| < \delta$, then



$|x| = |(x - x_0) + x_0| \leq |x - x_0| + |x_0|$

$< |x_0| + |x_0| = 2|x_0|$

$|x^n - x_0^n| = |x - x_0| \left| \sum_{j=0}^{n-1} x^j x_0^{n-1-j} \right|$

$\leq |x - x_0| \left(\sum_{j=0}^{n-1} |x^j x_0^{n-1-j}| \right)$ $\triangle$ ineq.

$= |x - x_0| \left(\sum_{j=0}^{n-1} \underbrace{|x^j|}_{2|x_0|} |x_0^{n-1-j}| \right)$

$\leq |x - x_0| \sum_{j=0}^{n-1} \underbrace{(2|x_0|)^j x_0^{n-1-j}}$

$= |x - x_0| \left(\sum_{j=0}^{n-1} 2^j |x_0|^{n-1} \right) < |x - x_0| \left(\sum_{j=0}^{n-1} 2^n |x_0|^{n-1} \right)$

$= |x - x_0| n \cdot 2^n |x_0|^{n-1} \leq \frac{\varepsilon}{2^n n |x_0|^{n-1}} n 2^n |x_0|^{n-1} = \varepsilon$ ∎

$(x - x_0)(x^{n-1} + x^{n-2} x_0 + x^{n-3} x_0^2 + \dots x_0^{n-1})$

$= (x^n + x^{n-1} x_0 + x^{n-2} x_0^2 + \dots + x x_0^{n-1})$

$(-x^{n-1} x_0 - x^{n-2} x_0^2 + \dots)$

$= x^n - x_0^n$ (all cancelled)

<u>Case 2</u>: $x_0 = 0$ Put $\delta = \sqrt[n]{\varepsilon}$. if $|x - 0| < \sqrt[n]{\varepsilon}$,
then $|x^n - 0| < (\sqrt[n]{\varepsilon})^n = \varepsilon$ ∎

$f(x) = x^n$ continued...

true for all n; induction

$P(n)$ a statement ab positive integers

if 1.) $P(1)$ is true

2.) $\forall K$, $P(K) \Rightarrow P(K+1)$

Then $P(n)$ is true for all pos. int.

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$, Prove f is continuous

($P(n)$ is "$f(x) = x^n$" is continuous)

Pf: let $f_n(x) = x^n$

1.) show $f_1(x)$ is continuous

let $x_0 \in \mathbb{R}$, let ε be given. Put $\delta = \varepsilon$.

if $|x - x_0| < \delta$, then $|f_1(x) - f_1(x_0)| = |x - x_0| < \delta = \varepsilon$

Induct! 2.) Assume that f_{n-1} is cont. (and $n \geq 2$)

let $x_0 \in \mathbb{R}$ and let ε be given

Put $\delta_1 = \frac{\varepsilon}{2(|x_0|^{n-1}+1)}$ choose δ_2 so that if

$|x - x_0| < \delta_2$, then $|x^{n-1} - x_0^{n-1}| < \frac{\varepsilon}{2(|x_0|+1)}$

(possible since f_{n-1} is cont.)

Put $\delta = \min\{1, \delta_1, \delta_2\}$

if $|x - x_0| < \delta$, then $|x^n - x_0^n| =$

$|x^{n-1} \cdot x - x_0^{n-1} x + x_0^{n-1} x - x_0^n|$

(factor) $|(x^{n-1} - x_0^{n-1})x + x_0^{n-1}(x - x_0)|$

$\leq |(x^{n-1} - x_0^{n-1})x| + |x_0^{n-1}(x - x_0)|$

$= |x^{n-1} - x_0^{n-1}|\,|x| + |x_0|^{n-1}\,|x - x_0|$ ← since $|x - x_0| < \delta_1$

since $|x - x_0| < \delta_2$ → $< \frac{\varepsilon}{2(|x_0|+1)}(|x_0|+1) + |x_0|^{n-1} \frac{\varepsilon}{2(|x_0^{n-1}|+1)}$

since $|x - x_0| < 1$ $= \frac{\varepsilon}{2} + \underbrace{\frac{|x_0|^{n-1}}{|x_0|^{n-1}+1}}_{<1} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

3. "A mathematician's Approach"

Thm: A product of finitely many continuous functions is continuous.
[if $f_1, f_2, f_3, \dots f_n$ are cont., $f_1 \cdot f_2 \cdot f_3 \cdots f_n$ is cont.]

Pf: Consider first a product of two cont. functions f and g.
now let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ be given.
Choose δ_1 so that if $|x-x_0| < \delta_1$, then $|g(x)-g(x_0)| < 1$
Choose $\delta_2 > 0$ " " if $|x-x_0| < \delta_2$, then $|f(x)-f(x_0)| < \frac{\varepsilon}{2(|g(x_0)|+1)}$
" $\delta_3 > 0$ so that if $|x-x_0| < \delta_3$, then
$|g(x)-g(x_0)| < \frac{\varepsilon}{2(|f(x_0)|+1)}$

if $|x-x_0| < \min\{\delta_1, \delta_2, \delta_3\}$, then

$$|f(x)g(x) - f(x_0)g(x_0)|$$
$$= |f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)|$$
$$\overset{\triangle \text{ ineq.}}{\le} |f(x) - f(x_0)||g(x)| + |f(x_0)||g(x) - g(x_0)|$$
$$< \frac{\varepsilon}{2(|g(x_0)|+1)}(|g(x_0)|+1) + |f(x_0)| \frac{\varepsilon}{2(|f(x_0)|+1)}$$
$$= \frac{\varepsilon}{2} + \frac{f(x_0)^{<1}}{f(x_0)+1} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Inductively, assume that a product of fewer than n cont. functions is continuous, and assume that $f_1, f_2, \dots, f_n$ are continuous
$\prod_{i=1}^{n} f_i = \left(\prod_{i=1}^{n-1} f_i\right) f_n$ By induction, the product of the first $n-1$ is cont.
so (product)(product) that is a product of two cont. functions, so their product is cont.

Original Problem:
Corollary: the function f defined by $f(x) = x^n$ is continuous
Pf: let $f_n(x) = x^n$. f_1 is cont. since if $|x-x_0| < \varepsilon$, then $|f_1(x) - f_1(x_0)| = |x-x_0| < \varepsilon$
$f_n = \prod_{i=1}^{n} f_1$. By the Thm, f_n is continuous.

$$\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0) = \frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}$$

HOMEWORK

$$= \frac{f(x)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)}$$

close to $g(x_0)$ when x is close to x_0

$$\frac{(f(x) - f(x_0))g(x_0) + f(x_0)(g(x_0) - g(x))}{g(x)g(x_0)}$$

II. Euclidean Space

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}\} \qquad \mathbb{R}^1 = \mathbb{R}$$

$\mathbb{R}$ $\qquad$ $\mathbb{R}^2$ $\qquad$ $\mathbb{R}^3$

Induct.

for $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$, define

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2 \dots + x_n^2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

- This is the **norm** of x, the distance from x to $(0, 0, \dots 0)$
- We define the distance from x to y by the norm of $\|x-y\|$
- In $\mathbb{R}^1$, $\|x\| = (x^2)^{1/2} = |x|$ — (the norm generalizes this)

$$\|x-y\| = \|(x_1 \cdots x_n) - (y_1 \cdots y_n)\| = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$$

- $\mathbb{R}^n$ with this choice of the distance function is called n-dimensional Euclidean Space.