Mathematics 3333-002 Name (please print) ______________________

Examination III Form A

April 28, 2010

Instructions: Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers, so if you find yourself involved in a lengthy calculation, it might be a good idea to move on and come back to that problem if you have time.

I. (6) Let A be the matrix

$$\begin{bmatrix} t & -2 & 0 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & t & 0 & 2 \\ -t & 0 & 3 & 4 \end{bmatrix}.$$

(a) Calculate $\det(A)$ as follows. First do an elementary row operation to make the $(4,1)$-entry equal to 0, then do cofactor expansion down the first column to reduce to computing the determinant of a 3×3 matrix. On that 3×3 matrix, do an elementary row operation that creates a second 0 in the middle column, and continue from there.

Performing $R_4 + R_1 \to R_4$, we obtain the matrix

$$\begin{bmatrix} t & -2 & 0 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & t & 0 & 2 \\ 0 & -2 & 3 & 1 \end{bmatrix}.$$

Now expand down the first column, and continue:

$$\begin{vmatrix} t & -2 & 0 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & t & 0 & 2 \\ 0 & -2 & 3 & 1 \end{vmatrix} = t \begin{vmatrix} 1 & 1 & 2 \\ t & 0 & 2 \\ -2 & 3 & 1 \end{vmatrix} = t \begin{vmatrix} 1 & 1 & 2 \\ t & 0 & 2 \\ -5 & 0 & -5 \end{vmatrix} = -t \begin{vmatrix} t & 2 \\ -5 & -5 \end{vmatrix} = -t(-5t+10) = 5t^2 - 10t$$

(b) Using your expression for $\det(A)$, find the values of t for which A is singular.

Solving $0 = \det(A) = 5t^2 - 10t = 5t(t-2)$ gives the values $t = 0$ and $t = 2$.

II. (10) Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b-c \\ 2b+c \\ -2a+3c \end{bmatrix}.$$

(You do not need to verify that L is linear.) As you know, the standard matrix representation of L is

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ -2 & 0 & 3 \end{bmatrix}.$$

(a) Use the standard matrix representation to find a basis for the kernel of L.

The kernel is the null space of A, that is, the space of solutions of $AX = 0$. We find it using elementary row operations:

$$A \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

so the null space is $\begin{bmatrix} 3r/2 \\ -r/2 \\ r \end{bmatrix}$ and possible bases include $\left\{\begin{bmatrix} 3/2 \\ -1/2 \\ 1 \end{bmatrix}\right\}$ or $\left\{\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}\right\}$.

(b) Use the standard matrix representation to find a basis for the range of L.

The range of L is the column space of A, which we find using elementary column operations:

$$A \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 0 \end{bmatrix}$$

and a basis is $\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$

III. (6) Let P be a nonsingular $n \times n$ matrix.

(a) Verify that $\det(P^{-1}) = 1/\det(P)$.

We have $1 = \det(I_n) = \det(PP^{-1}) = \det(P)\det(P^{-1})$, so $\det(P^{-1}) = 1/\det(P)$.

(b) Use part (a) to verify that if A is any $n \times n$ matrix, then $\det(P^{-1}AP) = \det(A)$.

$\det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = (1/\det(P))\det(A)\det(P) = \det(A)$.

IV. (10) Let P_2 be the space of polynomials of degree at most 2, and let S be the ordered basis $\{t^2-t+1, t-1, t^2+1\}$ of P_2.

(a) If the S-coordinate vector of the polynomial p is $p_S = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, find p.

$p = -(t^2 - t + 1) + 2(t - 1) + (t^2 + 1) = 3t - 2$.

(b) Find the S-coordinate vector of the polynomial $3t^2 - 2t + 4$.

We solve

$$3t^2 - 2t + 4 = a(t^2 - t + 1) + b(t - 1) + c(t^2 + 1) = (a + c)t^2 + (-a + b)t + (a - b + c)$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ -1 & 1 & 0 & -2 \\ 1 & -1 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

so $(a, b, c) = (1, -1, 2)$ and $(t^2 + t - 1)_S = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Check: $(t^2 - t + 1) - (t - 1) + 2(t^2 + 1) = 3t^2 - 2t + 4$.

(c) Let T be the basis $\{t^2, t, 1\}$ of P_2. Find the transition matrix (also called the change-of-basis matrix) $P_{T \leftarrow S}$ from S-coordinates to T-coordinates.

The columns of $P_{T \leftarrow S}$ are $(t^2 - t + 1)_T$, $(t - 1)_T$, and $(t^2 + 1)_T$, so

$$P_{T \leftarrow S} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

V. (8) (a) Let A be an $n \times m$ matrix, and let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be the matrix transformation defined by $L(v) = Av$. Verify that L is linear.

$L(av + bw) = A(av + bw) = A(av) + A(bw) = a\,Av + b\,Aw = a\,L(v) + b\,L(w)$.

(b) Let P_3 be the space of polynomials of degree at most 3, and let $L\colon P_3 \to P_3$ be the function defined by $L(p(t)) = p(t) + t$. By giving a specific counterexample, show that L is not linear.

$L(t + t) = t + t + t = 3t$, but $L(t) + L(t) = (t + t) + (t + t) = 4t$.

VI. (4) Let $A = ([a_{i,j}])$ be a 4×4 matrix, and consider the formula

$$\det(A) = \sum (\pm) a_{1,\sigma(1)} a_{2,\sigma(2)} a_{3,\sigma(3)} a_{4,\sigma(4)} \ .$$

Determine the sign (i. e. tell whether the term has a plus or a minus sign in the formula) of the term that contains $a_{1,2}a_{2,4}a_{3,3}a_{4,1}$ (make your reasoning clear— answers of "plus" or "minus" without a correct explanation won't receive any credit).

The permutation 2431 has $1 + 2 + 1 = 4$ inversions, so is even, so the term that contains $a_{1,2}a_{2,4}a_{3,3}a_{4,1}$ has a plus sign.

VII. (5) Let V be a vector space of dimension 3, and let $T = \{t_1, t_2, t_3\}$ be an ordered basis of V. Let $L\colon V \to V$ be the linear transformation whose matrix representation with respect to T-coordinates on the domain and the codomain is $A = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix}$. Write $L(t_1 + t_2 - 2t_3)$ as a linear combination of t_1, t_2, and t_3.

$$(L(t_1 + t_2 - 2t_3))_T = A\ (t_1 + t_2 - 2t_3)_T = A \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix}, \text{ so } L(t_1 + t_2 - 2t_3) = -4t_1 + t_2 + 2t_3.$$

VIII. (4) An $n \times n$ matrix B is obtained from a matrix $A = [a_{i,j}]$ by the elementary row operation $kR_i \to R_i$. Use the formula for $\det(A)$ to explain why $\det(B) = k\det(A)$.

The formula for the determinant gives

$$\det(B) = \sum (\pm) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots (k a_{i,\sigma(i)}) \cdots a_{n,\sigma(n)} = \sum (\pm) k a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{i,\sigma(i)} \cdots a_{n,\sigma(n)}$$
$$= k \sum (\pm) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{i,\sigma(i)} \cdots a_{n,\sigma(n)} = k\det(A) \ .$$