I. Let $A$ be the matrix

$$A = \begin{bmatrix}
t & -2 & 0 & -3 \\
0 & 1 & 1 & 2 \\
0 & t & 0 & 2 \\
-t & 0 & 3 & 4
\end{bmatrix}.$$ 

(a) Calculate $\det(A)$ as follows. First do an elementary row operation to make the $(4,1)$-entry equal to 0, then do cofactor expansion down the first column to reduce to computing the determinant of a $3 \times 3$ matrix. On that $3 \times 3$ matrix, do an elementary row operation that creates a second 0 in the middle column, and continue from there.

(b) Using your expression for $\det(A)$, find the values of $t$ for which $A$ is singular.

II. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$L \begin{pmatrix}
a \\
b \\
c
\end{pmatrix} = \begin{pmatrix}
a + b - c \\
2b + c \\
-2a + 3c
\end{pmatrix}.$$ 

(You do not need to verify that $L$ is linear.) As you know, the standard matrix representation of $L$ is

$$A = \begin{bmatrix}
1 & 1 & -1 \\
0 & 2 & 1 \\
-2 & 0 & 3
\end{bmatrix}.$$ 

(a) Use the standard matrix representation to find a basis for the kernel of $L$.

(b) Use the standard matrix representation to find a basis for the range of $L$.

III. Let $P$ be a nonsingular $n \times n$ matrix.

(a) Verify that $\det(P^{-1}) = 1/\det(P)$.

(b) Use part (a) to verify that if $A$ is any $n \times n$ matrix, then $\det(P^{-1}AP) = \det(A)$. 
IV. Let $P_2$ be the space of polynomials of degree at most 2, and let $S$ be the ordered basis \{t^2 - t + 1, t - 1, t^2 + 1\} of $P_2$.

(a) If the $S$-coordinate vector of the polynomial $p$ is $p_S = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, find $p$.

(b) Find the $S$-coordinate vector of the polynomial $3t^2 - 2t + 4$.

(c) Let $T$ be the basis \{t^2, t, 1\} of $P_2$. Find the transition matrix (also called the change-of-basis matrix) $P_{T \rightarrow S}$ from $S$-coordinates to $T$-coordinates.

V. (a) Let $A$ be an $n \times m$ matrix, and let $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the matrix transformation defined by $L(v) = Av$. Verify that $L$ is linear.

(b) Let $P_3$ be the space of polynomials of degree at most 3, and let $L: P_3 \rightarrow P_3$ be the function defined by $L(p(t)) = p(t) + t$. By giving a specific counterexample, show that $L$ is not linear.

VI. Let $A = ([a_{i,j}])$ be a $4 \times 4$ matrix, and consider the formula

$$\det(A) = \sum (\pm)a_{1,\sigma(1)}a_{2,\sigma(2)}a_{3,\sigma(3)}a_{4,\sigma(4)}.$$ 

Determine the sign (i.e., tell whether the term has a plus or a minus sign in the formula) of the term that contains $a_{1,2}a_{2,4}a_{3,3}a_{4,1}$ (make your reasoning clear—answers of “plus” or “minus” without a correct explanation won’t receive any credit).

VII. Let $V$ be a vector space of dimension 3, and let $T = \{t_1, t_2, t_3\}$ be an ordered basis of $V$. Let $L: V \rightarrow V$ be the linear transformation whose matrix representation with respect to $T$-coordinates on the domain and the codomain is $A = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix}$. Write $L(t_1 + t_2 - 2t_3)$ as a linear combination of $t_1$, $t_2$, and $t_3$.

VIII. An $n \times n$ matrix $B$ is obtained from a matrix $A = [a_{i,j}]$ by the elementary row operation $kR_i \rightarrow R_i$. Use the formula for $\det(A)$ to explain why $\det(B) = k\det(A)$.