Mathematics 3333-002 Name (please print) ______________________

Examination II Form A

March 24, 2010

Instructions: Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers, so if you find yourself involved in a lengthy calculation, it might be a good idea to move on and come back to that problem if you have time.

I. (8) Consider the vector space $V = \{(x, y, z) \mid x, y, z \text{ are in } \mathbb{R}\}$ with the operations $(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')$ and $\lambda \odot (x, y, z) = (x, 1, z)$.

(a) Verify that V satisfies the vector space axiom $\lambda \odot (\mu \odot v) = (\lambda\mu) \odot v$. [Hint: write v as (x, y, z).]

For any numbers λ and μ, and any (x, y, z) in V,

$$\lambda \odot (\mu \odot (x, y, z)) = \lambda \odot (x, 1, z) = (x, 1, z)$$

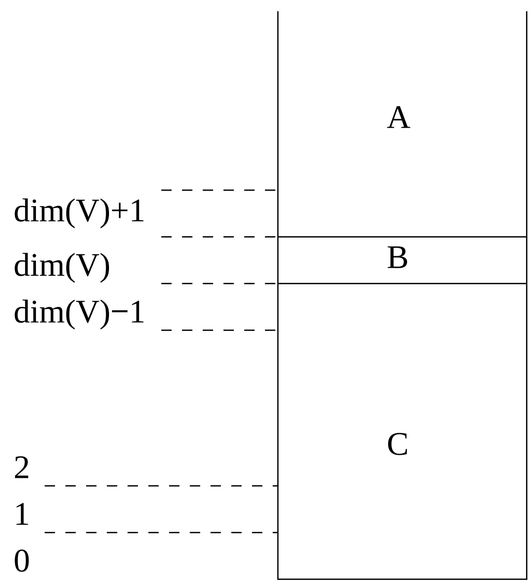
and

$$(\lambda\mu) \odot (x, y, z) = (x, 1, z) \ .$$

(b) Tell one of the eight vector space axioms that V fails to satisfy, and verify that V fails to satisfy it.

Possible choices are
$c \odot (v \oplus w) = (c \odot v) \oplus (c \odot w)$ [sample counterexample: $1 \odot ((1, 1, 1) \oplus (1, 1, 1)) = 1 \odot (2, 2, 2) = (2, 1, 2)$ but $(1 \odot (1, 1, 1)) \oplus (1 \odot (1, 1, 1)) = (1, 1, 1) \oplus (1, 1, 1) = (2, 2, 2)$],
$(c + d) \odot u = (c \odot u) \oplus (d \odot u)$ [sample counterexample: $(1 + 1) \odot (1, 1, 1) = 2 \odot (1, 1, 1) = (1, 1, 1)$ but $(1 \odot (1, 1, 1)) \oplus (1 \odot (1, 1, 1)) = (1, 1, 1) + (1, 1, 1) = (2, 2, 2)$]
$1 \odot u = u$ [sample counterexample: $1 \odot (2, 2, 2) = (2, 1, 2) \neq (2, 2, 2)$]

II. (5) The regions A, B, and C in this diagram represent *some* of the finite subsets of the finite subsets of a vector space V, specifically *those which either span, or are linearly independent, or both.* The number of elements in the subsets is indicated by the numbers to the left, 0, 1, 2, ..., $\dim(V) - 1$, $\dim(V)$, $\dim(V) + 1, \ldots$ and so on. For each of the following, tell which region or regions comprise the subsets that:



(a) are linearly independent [B and C]

(b) span [A and B]

(c) are bases [B]

III. Using the definition of linear independence, verify that the set $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ is linearly independent.
(5)

Suppose that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\lambda_1 - 2\lambda_2 \\ 3\lambda_1 + 3\lambda_2 \end{bmatrix} .$$

Solving this linear system by Gauss-Jordan elimination gives

$$\begin{bmatrix} 2 & -2 & 0 \\ 3 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 3 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 6 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

so the only solution is $\lambda_1 = \lambda_2 = 0$.

IV. Define what it means to say that a subset $\{v_1, v_2, \ldots, v_n\}$ of a vector space V is a *basis.* Define the *dimension* of V.
(4)

To say that the subset $\{v_1, v_2, \ldots, v_n\}$ is a basis means that it spans V and is linearly independent. The dimension of V is the number of elements in a basis.

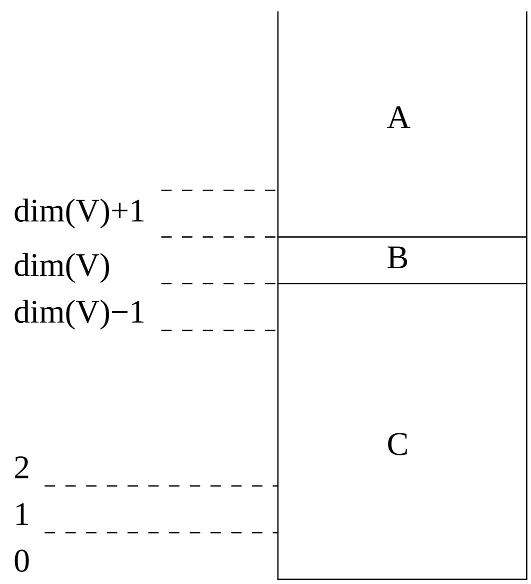
V. The regions A, B, and C in this diagram represents *all* of the finite subsets of a vector space V. The number of elements in the subsets is indicated by the numbers to the left, 0, 1, 2, …, $\dim(V) - 1$, $\dim(V)$, $\dim(V) + 1, \ldots$ and so on. For each of the following, tell which region or regions comprise the subsets that:
(6)

(a) might span [A and B]

(b) cannot span [C]

(c) might be linearly independent [B and C]

(d) cannot be linearly independent [A]



VI. (5) Recall that if $\{v_1, \ldots, v_k\}$ is a subset of a vector space V, then the span of $\{v_1, \ldots, v_k\}$ is $\text{span}(\{v_1, \ldots, v_k\}) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid \text{ the } \lambda_i \text{ are numbers.}\}$. Verify that $\text{span}(\{v_1, \ldots, v_k\})$ is closed under addition and scalar multiplication.

Take any $\lambda_1 v_1 + \cdots + \lambda_k v_k$ and $\mu_1 v_1 + \cdots + \mu_k v_k$ in $\text{span}(\{v_1, \ldots, v_k\})$. Adding these gives $(\lambda_1 + \mu_1)v_1 + \cdots + (\lambda_k + \mu_k)v_k$, which is also in $\text{span}(\{v_1, \ldots, v_k\})$. Also, for any number λ, $\lambda(\lambda_1 v_1 + \cdots + \lambda_k v_k) = (\lambda\lambda_1)v_1 + \cdots + (\lambda\lambda_k)v_k$, which is again in $\text{span}(\{v_1, \ldots, v_k\})$.

VII. (8) Find a basis for and the dimension of the solution space of this homogeneous system:

$$\begin{bmatrix} 1 & 3 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using back substitution to write down the solutions, the free parameters correspond to $x_2 = a$, $x_3 = b$, and $x_5 = c$, and we have $x_4 = 5x_5 = 5c$ and $x_1 = -3x_2 + 5x_5 = -3a + 5c$, so the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3a + 5c \\ a \\ b \\ 5c \\ c \end{bmatrix}.$$

By the usual method of putting one parameter equal to 1 and the others 0, we obtain the basis

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

The dimension of the solution space is 3.

VIII. Find a basis for the row space of the matrix $\begin{bmatrix} 3 & 0 & 1 \\ 3 & -3 & 3 \\ -2 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$.
(6)

Putting it in row echelon form, we have

$$\begin{bmatrix} 3 & 0 & 1 \\ 3 & -3 & 3 \\ -2 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & -3 & 2 \\ 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -2/3 \\ 1 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So a basis for the row space is $\left\{ \begin{bmatrix} 1 & 0 & 1/3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2/3 \end{bmatrix} \right\}$.

IX. (a) The rank of a certain 5×4 matrix A is 2. What is the dimension of the solution space of the homogeneous linear system $AX = 0$? Why?
(8)

The dimension of the solution space is 2. Since A is 5×4, the number of variables is $n = 4$. The rank is given as 2, so the nullity must be $4 - 2 = 2$. That is, the dimension of the solution space of the homogeneous linear system $AX = 0$ is 2.

(b) A certain matrix B is the coefficient matrix of a homogeneous linear system of five equations. If the dimension of the solution space of the homogeneous linear system $BX = 0$ is 5, and the dimension of the column space of B is 3, how many variables does the linear system have? Why?

There are eight variables. The dimensions of B are $5 \times n$, where n is the number of variables. The nullity of B is 5. The rank of B is the same as the dimension of the column space, which is 3. Since n is the rank plus the nullity, there must be eight variables.