I. Consider the vector space \( V = \{(x, y, z) \mid x, y, z \text{ are in } \mathbb{R}\} \) with the operations \((x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')\) and \(\lambda \odot (x, y, z) = (x, 1, z)\).

(a) Verify that \( V \) satisfies the vector space axiom \( \lambda \odot (\mu \odot v) = (\lambda \mu) \odot v. \) [Hint: write \( v \) as \((x, y, z)\).]

For any numbers \(\lambda\) and \(\mu\), and any \((x, y, z)\) in \(V\),

\[ \lambda \odot (\mu \odot (x, y, z)) = \lambda \odot (x, 1, z) = (x, 1, z) \]

and

\[ (\lambda \mu) \odot (x, y, z) = (x, 1, z) \cdot \]

(b) Tell one of the eight vector space axioms that \( V \) fails to satisfy, and verify that \( V \) fails to satisfy it.

Possible choices are
\[ c \odot (v \oplus w) = (c \odot v) \oplus (c \odot w) \] [sample counterexample: \( 1 \odot ((1, 1, 1) \oplus (1, 1, 1)) = 1 \odot (2, 2, 2) = (2, 1, 2) \)
but \((1 \odot (1, 1, 1)) \oplus (1 \odot (1, 1, 1)) = (1, 1, 1) \oplus (1, 1, 1) = (2, 2, 2)\),
\[ (c + d) \odot u = (c \odot u) \oplus (d \odot u) \] [sample counterexample: \( (1 + 1) \odot (1, 1, 1) = 2 \odot (1, 1, 1) = (1, 1, 1) \)
but \((1 \odot (1, 1, 1)) \oplus (1 \odot (1, 1, 1)) = (1, 1, 1) \oplus (1, 1, 1) = (2, 2, 2)\)
\[ 1 \odot u = u \] [sample counterexample: \( 1 \odot (2, 2, 2) = (2, 1, 2) \neq (2, 2, 2)\)]

II. The regions \(A, B,\) and \(C\) in this diagram represent some of the finite subsets of the finite subsets of a vector space \(V\), specifically those which \(either\) span, \(or\) are linearly independent, \(or\) both.

The number of elements in the subsets is indicated by the numbers to the left, \(0, 1, 2, \ldots\), \(\dim(V) - 1, \dim(V), \dim(V) + 1, \ldots\) and so on.

For each of the following, tell which region or regions comprise the subsets that:

(a) are linearly independent \([B\ and\ C]\)

(b) span \([A\ and\ B]\)

(c) are bases \([B]\)
III. Using the definition of linear independence, verify that the set \( \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix} \) is linearly independent.

Suppose that
\[
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\lambda_1 - 2\lambda_2 \\ 2\lambda_1 + 3\lambda_2 \end{bmatrix}.
\]

Solving this linear system by Gauss-Jordan elimination gives
\[
\begin{bmatrix} 2 & -2 & 0 \\ 3 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 3 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\]
so the only solution is \( \lambda_1 = \lambda_2 = 0 \).

IV. Define what it means to say that a subset \( \{v_1, v_2, \ldots, v_n\} \) of a vector space \( V \) is a basis. Define the dimension of \( V \).

To say that the subset \( \{v_1, v_2, \ldots, v_n\} \) is a basis means that it spans \( V \) and is linearly independent. The dimension of \( V \) is the number of elements in a basis.

V. The regions A, B, and C in this diagram represents all of the finite subsets of a vector space \( V \). The number of elements in the subsets is indicated by the numbers to the left, 0, 1, 2,\ldots, \( \dim(V) - 1 \), \( \dim(V) \), \( \dim(V) + 1 \),\ldots and so on. For each of the following, tell which region or regions comprise the subsets that:

(a) might span [A and B]
(b) cannot span [C]
(c) might be linearly independent [B and C]
(d) cannot be linearly independent [A]
VI. Recall that if \( \{v_1, \ldots, v_k\} \) is a subset of a vector space \( V \), then the span of \( \{v_1, \ldots, v_k\} \) is \( \text{span}(\{v_1, \ldots, v_k\}) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid \text{ the } \lambda_i \text{ are numbers.} \} \). Verify that \( \text{span}(\{v_1, \ldots, v_k\}) \) is closed under addition and scalar multiplication.

Take any \( \lambda_1 v_1 + \cdots + \lambda_k v_k \) and \( \mu_1 v_1 + \cdots + \mu_k v_k \) in \( \text{span}(\{v_1, \ldots, v_k\}) \). Adding these gives \( (\lambda_1 + \mu_1) v_1 + \cdots + (\lambda_k + \mu_k) v_k \), which is also in \( \text{span}(\{v_1, \ldots, v_k\}) \). Also, for any number \( \lambda \), \( \lambda(\lambda_1 v_1 + \cdots + \lambda_k v_k) = (\lambda \lambda_1) v_1 + \cdots + (\lambda \lambda_k) v_k \), which is again in \( \text{span}(\{v_1, \ldots, v_k\}) \).

VII. Find a basis for and the dimension of the solution space of this homogeneous system:

\[
\begin{bmatrix}
1 & 3 & 0 & 0 & -5 \\
0 & 0 & 1 & -5 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Using back substitution to write down the solutions, the free parameters correspond to \( x_2 = a \), \( x_3 = b \), and \( x_5 = c \), and we have \( x_4 = 5x_5 = 5c \) and \( x_1 = -3x_2 + 5x_5 = -3a + 5c \), so the general solution is

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= 
\begin{bmatrix}
-3a + 5c \\
a \\
b \\
5c \\
c
\end{bmatrix}
\]

By the usual method of putting one parameter equal to 1 and the others 0, we obtain the basis

\[
\begin{pmatrix}
-3 & 0 & 5 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 5 \\
0 & 0 & 1
\end{pmatrix}
\]

The dimension of the solution space is 3.
VIII. Find a basis for the row space of the matrix
\[
\begin{bmatrix}
3 & 0 & 1 \\
3 & -3 & 3 \\
-2 & -1 & 0 \\
1 & -1 & 1
\end{bmatrix}.
\]

Putting it in row echelon form, we have
\[
\begin{bmatrix}
3 & 0 & 1 \\
3 & -3 & 3 \\
-2 & -1 & 0 \\
1 & -1 & 1
\end{bmatrix} \rightarrow
\begin{bmatrix}
0 & 3 & -2 \\
0 & 0 & 0 \\
0 & 1 & -2/3 \\
0 & 0 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 0 & 1/3 \\
0 & 1 & -2/3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

So a basis for the row space is \{ \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -2/3 \end{bmatrix} \}.

IX. (a) The rank of a certain 5 \times 4 matrix A is 2. What is the dimension of the solution space of the homogeneous linear system \( AX = 0 \)? Why?

The dimension of the solution space is 2. Since \( A \) is 5 \times 4, the number of variables is \( n = 4 \). The rank is given as 2, so the nullity must be \( 4 - 2 = 2 \). That is, the dimension of the solution space of the homogeneous linear system \( AX = 0 \) is 2.

(b) A certain matrix \( B \) is the coefficient matrix of a homogeneous linear system of five equations. If the dimension of the solution space of the homogeneous linear system \( BX = 0 \) is 5, and the dimension of the column space of \( B \) is 3, how many variables does the linear system have? Why?

There are eight variables. The dimensions of \( B \) are 5 \times \( n \), where \( n \) is the number of variables. The nullity of \( B \) is 5. The rank of \( B \) is the same as the dimension of the column space, which is 3. Since \( n \) is the rank plus the nullity, there must be eight variables.